# Rhombopolyclonic Polygonal Rosettes Theory

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**Abstract**
The division of a regular polygon with an even numbers of vertices into a whole number of “isoperimetric” rhombuses (equal sides and different angles) is possible. Elementary reasoning leads to a general theory, which offers many possibilities of plastic applications.

## 1. Introduction

The following natural idea occurred to me. I started by drawing a fractal figure, dividing up a regular dodecagon into triangles (Figure 1):

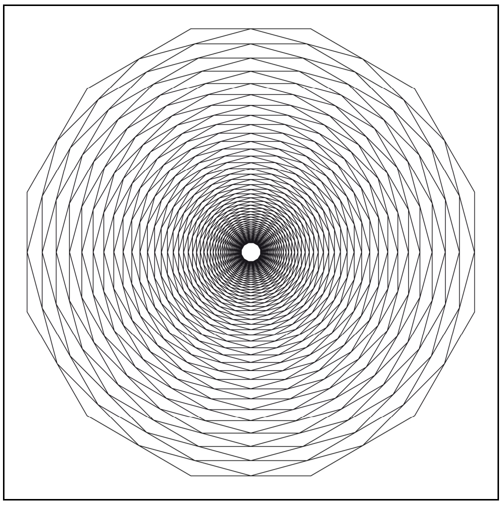

**Fig.1. Fractal octodecagon**

Colouring this drawing, I obtained a picture that is reminiscent of the diffraction of light (Figure 2):

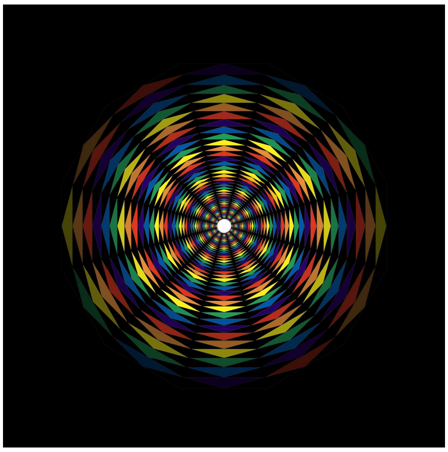

**Fig.2. “Diffraction of light”**

. However, I found this picture on the cover of a publication dedicated to the golden ration. Therefore, I took no further interest in this particular example. I asked myself what would happen if rather than divide the inside of a polygon into triangles I instead divided it into rhombuses.

Let *p* denote the length of the of an edge of the polygon and *c* denote the length of the edge of a rhombus.

I first tried to divide the polygon into successive rings of isoperimetric rhombuses starting from the periphery and extending towards the centre.

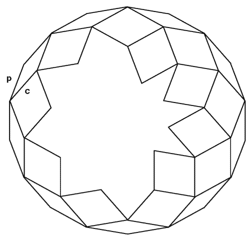


**Fig.3. Starting from the periphery with c = p**

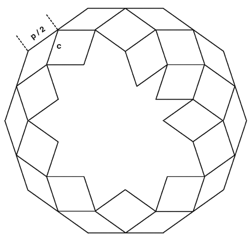


**Fig.4. Starting from the periphery with c = p/2**

: In either case, it is not clear what is going to happen when we approach the centre of the polygon.
The idea occurred to me to invert the process, starting from the centre of the polygon.

**2. Construction of rhomboid polygons from a regular "polyclon"**
Given a rhombus whose vertices are (A,B,C,D), whose edges are (AB), (BC), (CD), (DA), its diagonals are (A,C) and (B,D). The angle between the edges (AB) and (AD) will be denoted by $\alpha$, the angle between the edges (BA) and (BC) by $\beta$ ($\beta = \pi - \alpha$).

We denote by R(c, $\alpha$) a rhombus whose edge length is c, and which has an interior angle of $\alpha$.

n being an integer, we suppose $n \geq 3$, $n \neq 4$.

**Definitions :**
1) Two rhombuses R(c, $\alpha$) and R(c', $\alpha$') are *isoperimetric* if c = c'.
2) Given a regular convex n-polygon (n is its number of edges), a *n-ring* of isoperimetric rhombuses over the given n-polygon consists in the data of n isoperimetric rhombuses whose one diagonal of each one is an edge of the polygon.
3) If the centre O of the polygon is a common vertex to all the rhombuses, the n-ring will be said to be of *rank* 1, the polygon will be called a *polyclon*, or a *basic polygon*, or a *rank 1 polygon*.

Here is an example of a rank 1 9-ring (Fig. 5).

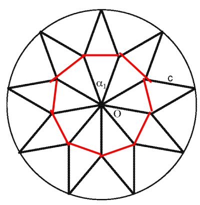


**Fig.5. Enneaclon (a rank 1 9-ring)**

Starting from a rank 1 n-polygon, it is easy to construct a series of new polygons.

**Example :** Let us start with the rank 1 9-ring of figure 5. We construct a second rank 9-ring in the following way (see fig 6) : the n = 9 vertices of this ring which are opposite to the centre O of the basic polygon define a new regular n-polygon, which is a rank 2 9-polygon over the basic polygon. The n = 9 vertices of the rank 2 9-polygon will be named $V_{2i}$ (i = 1, 2, …, n) and will called the vertices of second rank. The n = 9 edges of the second rank regular n-polygon are the diagonals of n = 9 new rhombuses which are isoperimetric with the previous ones. They form a rank 2 n = 9-ring of rhombuses.

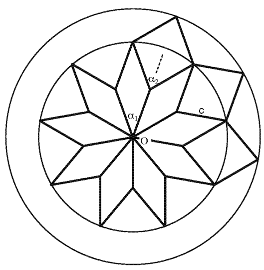

**Fig.6. Beginning of the construction of a rank 2 9-ring of rhombuses**

The n = 9 vertices $V_{1i}$ (i = 1, 2, …, n) of the rhombuses of the first ring which are not opposite to O, will be called *first rank vertices*.
If $\alpha_{1n}$ is the angle of a rhombus in O, the angle $\alpha_{2n}$ in $V_{1i}$ of a rhombus from the second ring is given by $\alpha_{2n} = 2\ \alpha_{1n}$.

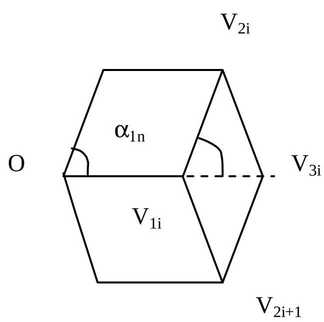


**Fig. 7.**

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Note that $\alpha_{1n} = 2\ \pi/n$, but since the previous relation does not depend on n, we shall drop the index n in the next relations between angles.
Constructing by the same process a third rank n = 9-ring of rhombuses, we observe that we get a convex regular 2xn = 2x9 = 18-polygon.

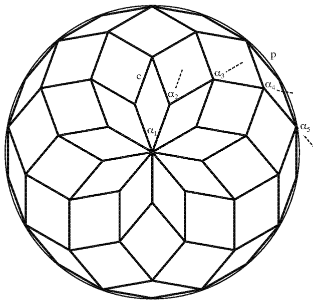

**Fig.8. Octodecagon obtained from an enneaclon**

Let $\alpha_3$ be the angle in $V_{2i}$ of a rhombus from the third ring. From the relations :

$$2(\pi - \alpha_2) + \alpha_1 + \alpha_3 = 2\pi$$
$$\alpha_2 = 2\ \alpha_1$$

we get :

$$\alpha_3 = 3\alpha_1$$

Suppose that now we add a fourth ring of rhombuses according to the previous process : the n vertices $V_{3i}$ on the rhombuses which are opposite to the vertices $V_{1i}$ define a rank 3 regular n-polygon from which a rank 3 of n rhombuses is constructed.
From similar relations to the previous ones:

$$2(\pi - \alpha_3) + \alpha_2 + \alpha_4 = 2\pi$$
$$\alpha_3 = 3\alpha_1$$
$$\alpha_2 = 2\ \alpha_1$$

one gets :

$$\alpha_4 = 4\ \alpha_1$$

More generally, by trivial induction, since $\alpha_1 = 2\ \pi/n$,

**Lemma :** $\alpha_k = k\,\alpha_1 = 2k\,\pi/n$.

In the example, $\alpha_1 = 2\,\pi/9$: when $k = 4$, $\alpha_4 = 8\,\pi/9$, while $\alpha_5 = 10\,\pi/9$
Since any interior angle of a rhombus is less than $\pi$, one cannot add in the present case a new ring of rhombuses, so that the process has to stop with the construction of the fourth ring of rhombuses.

**Definitions :** The adding of a ring of rhombuses to a basic polygon will be called a *partial expansion* of the polygon with rhombuses. When the process of creating such new rings has to stop, we shall say that the obtained polygon is the *complete expansion* of the basic polygon. It will be called a *rhomboid polygon*.

Note that any expansion generates a polygon with the same number 2n of edges.

**Corollary 1:** *The complete expansion of a basic regular* n-*polygon with rhombuses is a regular* 2n-*polygon consisting of* $r = [n/2] - 1$ *rings of rhombuses.*
Proof: Since $\alpha_k = k\,\alpha_1 = 2k\,\pi/n < \pi$, then $k < n/2$, so that its maximum value is $[n/2] - 1$ ,where $[n/2]$ is the integer part of $n/2$.

**Corollary 2:** *The former complete expansion is a convex polygon*.
Proof: The last two series of vertices of higher rank are the $V_{ri}$ and the $V_{r+1,i}$ vertices. To add a new rhombus, we have to catch one $V_{ri}$ vertex and two adjacent $V_{r+1,i}$ vertices. If the polygon was not convex in $V_{ri}$, then it would be possible to add the new rhombus. But since that is not possible, then the polygon has to be convex in $V_{ri}$. vex by construction at any $V_{r+1,i}$. Thus the complete expansion is convex at any vertex of its boundary.

Let us now look at the special case **when $n = 2\rho$** :
In that case, on one hand $r = \rho - 1$. On the other hand, let us consider any $V_{ri}$ vertex, and compute the sum of the angles of the three rhombuses having this vertex in common; setting $\alpha_1 = \alpha$, this sum is :

$$2(\pi - r\alpha) + (r-1)\,\alpha = 2\pi - (r+1)\alpha = \pi$$

since $$\alpha = 2\pi/n = \pi/(r+1)$$

Then :
**Proposition:** *Whenever n is even, the boundary of the complete expansion is a regular n-polygon whose edge has length* 2c.

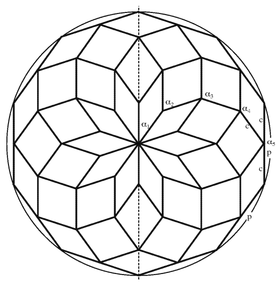

**Fig.9. Decagon obtained from a decagon**

When besides **n = 6m**, then for k = 3m/2, the k-ring is made of squares since k $\alpha$ = 3m/2 (2$\pi$/6m) = $\pi$/2.
Such a situation also occurs for the m-ring when **n = 4m**.

## 3. Dividing a regular n-polygon, n being an even integer

The following statement is a direct consequence of the results of the pervious paragraph, :

**Theorem 1 :** *Given any regular polygon with an even number* N *of vertices, there exists an infinite number of ways to divide it into isoperimetric rhombuses. Any given decomposition gives rise to an infinite associated series parametrized by the integers* $s^2$ *where* $s \in \mathbf{N}$.

Proof : Let N = 2n be the number of vertices of the polygon, p the length of an edge. The angle $\alpha = 2\pi/n = \pi/\rho$ and the value of p allow to define the main process of decomposition. There are two cases according to the parity of $\rho$.
When $\rho$ is odd, we can construct r = [$\rho$] – 1 rings of R(c, $\alpha$) rhombuses with c= p, by starting from the centre.
When $\rho$ is even, we can construct r = $\rho$ – 1 rings of R(c, $\alpha$) rhombuses with c = p/2 by starting from the centre.

Let K = r.n the numbers of rhombuses constructed for each case. It is now possible to divide each rhombus into s² isoperimetric rhombuses whose edge has length c/s where s is any integer.

**4. Picking up a rhombo-polygonal rosette on the sphere**

We shall only quote the main result :

**Theorem 2:** *Let* R *be a rhombo-polygonal rosette generated by a rank 1 n-ring where n is even. The successive rings of rhombuses on the plane are inscribed into circles which are the projections of parallels on the 2-sphere with the same angular distance.*

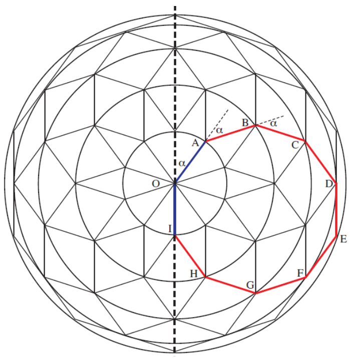


**Fig.10.**

Here is an image made for me by Los Leys :

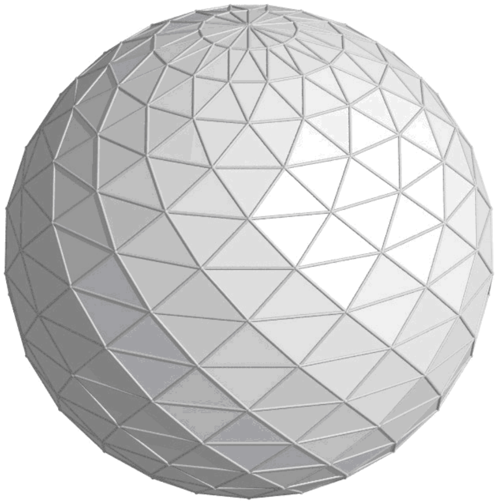

**Fig.11. Picking up on the sphere of a rank 1 18-ring**

## 5. Historical remarks and new works

Though the past literature does not seem to refer to the facts shown in the previous paragraphs, it is interesting to recall preliminary works by Dürer and Képler respectively on rosettes and on rhombuses.

When the number of vertices increases, any rhombopolyclonic rosette looks more and more like some Durer's drawing (Figure 9) [1].

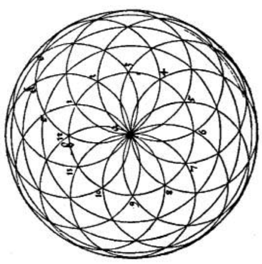

**Fig.12. Durer's rosette**

In that figure, we find arcs of circles instead of straight lines segments, and vertices of rhombus take place at the intersections of circles. Compare with the following figure :

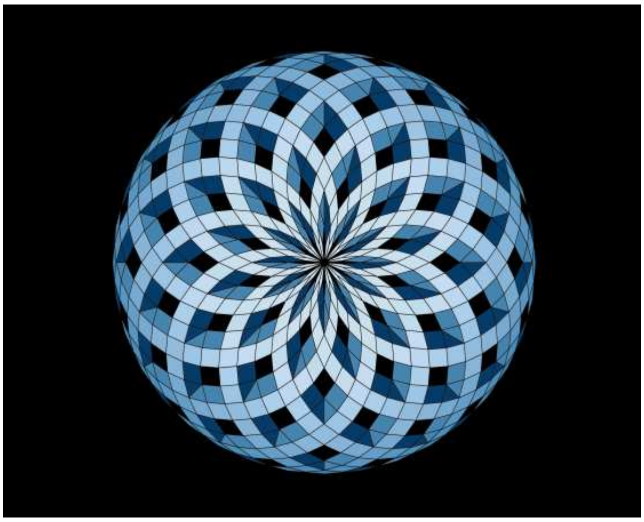

**Fig. 13. Big blue rosette**

Képler, studying the "thombic figure in the alveoles of bees" [2], got interested in constructing rhomboedric figures on the plane or in the 3-space. Here is a "pentarhomboclonic decagonal rosette" he has drawn (Figure 14) :

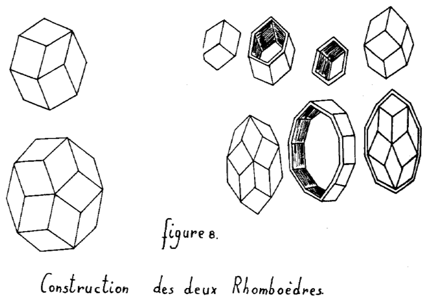


**Fig.14. Képler's drawing**

We shall conclude by showing two recent images :

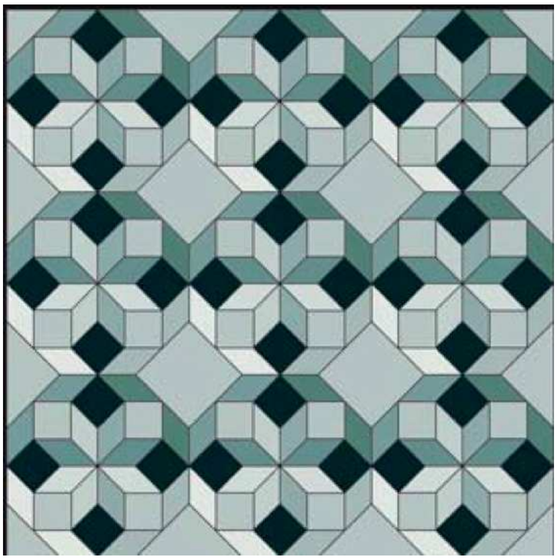

**Fig.15. Octorhomboclonic octagonal rosette**

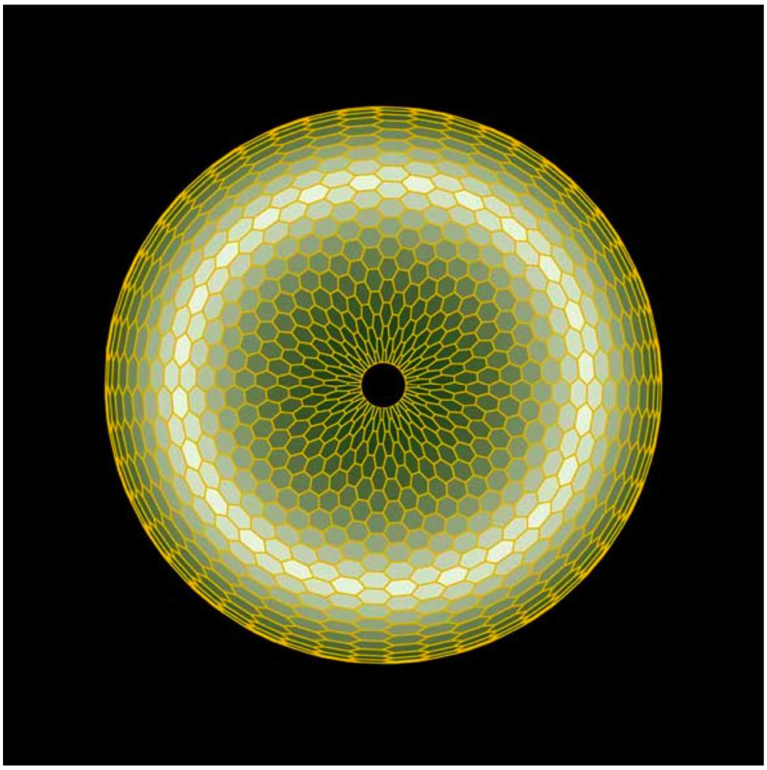

**Fig.16. Cyclic time snake**

## References

[1] DÜRER A., *Géométrie* (présentation et traduction J. Peiffer), Seuil, Paris, (1995).

[2] KEPLER J., *L'Etrenne ou la neige sexangulaire* (traduction et critique R. Halleux) CNRS-Vrin, Paris, (1975).

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