Geometry and Art from the Cordovan Proportion

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Abstract: The Cordovan proportion, $c=(2-\sqrt{2})^{-1/2}$, is the ratio between the radius of the regular octagon and its side length. This proportion was introduced by R. de la Hoz in 1973. Recently, the authors have found geometric properties linked with that proportion, related with a family of shapes named by them, *Cordovan polygons*. These results are summarized and are extended through the works of art of Hashim Cabrera and Luis Calvo, two Cordovan painters who have consciously considered the Cordovan proportion in their recent compositions. In fact, we have checked this ratio in several dissections of the canvases of Cabrera, and looking at the picture of Calvo, we have recognized many of our Cordovan polygons and some new polygons which we have added to our previous collection. We have also discovered some new cordovan dissections of a square, a $\sqrt{2}$ rectangle and a Silver rectangle.

1. Introduction

This work is the result of a meeting of two mathematicians with two painters in Córdoba city in Spain, and its conversations about the Cordovan proportion.

The Cordovan proportion, $c=(2-\sqrt{2})^{-1/2}$, is the ratio between the radius, R, of the regular octagon and its side length, L, (Figure 1).

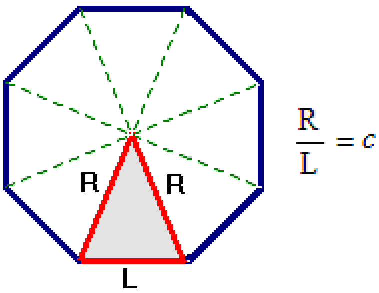

Fig. 1: Cordovan proportion in the regular octagon

This proportion was introduced by the Spanish architect Rafael de la Hoz Arderius in 1973 (Hoz, 1973, 2005) and named by him, *Cordovan proportion*. The irrational value of this ratio is known as the *Cordovan number*. By means of the law of Cosines, in the marked triangle with sides R, R and L (see Figure 1), we have:

$$L^2 = 2R^2 - 2R^2 \cos 45^\circ = R^2\left(2-\sqrt{2}\right) \Rightarrow \frac{R^2}{L^2} = \frac{1}{2-\sqrt{2}}$$

$$c = \frac{R}{L} = \frac{1}{\sqrt{2-\sqrt{2}}} = 1.306562964\ldots = Cordovan\ Number$$

Before 2008, when the authors began the study of this proportion, only the rectangular shape had been considered. From that year, their research has found geometric properties linked with that proportion, related to a wide family of new forms named by them, *Cordovan polygons* (Redondo and Reyes 2008a, 2008b, 2009).
In this contribution, those results are summarized and are extended through the works of art of two Cordovan painters which have consciously considered the Cordovan proportion in their recent compositions. The aims, motivations and visual results are completely different within their works.

2. Polygonal shapes and Cordovan proportion

In this section, we summarize the main polygonal shapes discovered and related with the Cordovan proportion.

The *"**Cordovan triangle**" is the isosceles triangle which is similar to the one in Figure 1 of sides R, R and L*. So, an isosceles triangle is a "Cordovan triangle" if its angles are π/4, 3π/8 and 3π/8 radians.

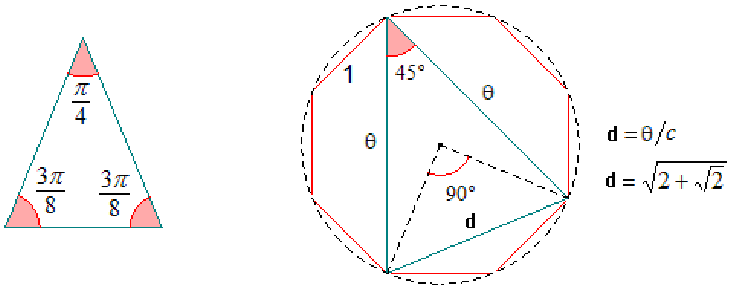

Fig. 2: Cordovan triangle and its location within the regular octagon.

The *"**Cordovan rectangle**"is a rectangle whose sides are in ratio c.* For instance, the marked rectangle of sides R and L showed in Figure 3.

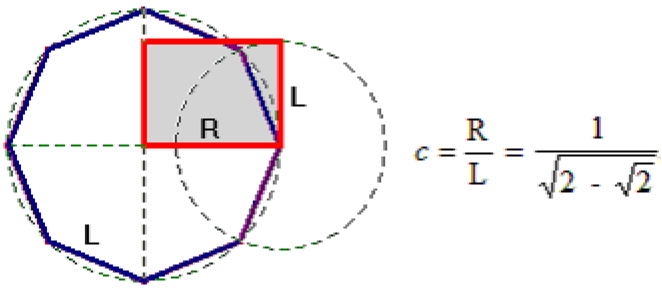

Fig. 3: Cordovan rectangle and its construction from a regular octagon.

The *"**Cordovan diamond**" is a rhombus whose angles are* π/4, and 3π/4 radians. This shape is formed by the union of two Cordovan triangles. Four octagons intersected as in Figure 4 to produce an inner star formed by four diamonds.

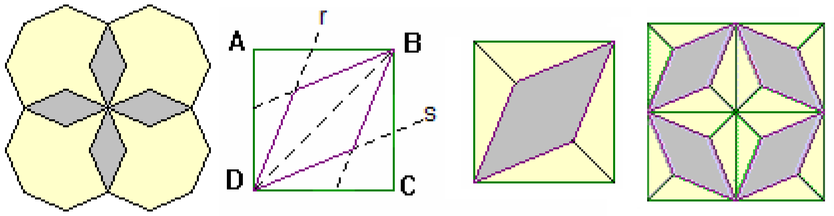

Fig. 4: Diamonds in octagons and Diamonds in a square by bisecting 90 and 45 degrees.

The regular octagon can be divided into four congruent quadrilaterals, named by the authors ***"Cordovan kites"*** *or* ***"c-kites"***. A *c*-kite is formed by two Cordovan triangles. This quadrilateral has angles π/2, 3π/8, 3π/4, and 3π/8. If the side of the octagon is 1, the *c*-kite has sides 1, 1, c, and c, Figure 5.

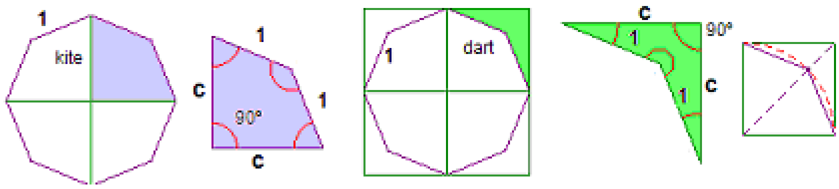

Figure 5: Cordovan kite, Cordovan dart and its construction from the square

When the octagon is inscribed in a square, another four congruent concave quadrilaterals (see Figure 5) appear. These quadrilaterals are called as *"**Cordovan darts**"or* ***"c-darts"***. A *c*-dart has angles π/8, π/2, π/8 and 5π/4, and sides 1, 1, c, and c.

Several ***Cordovan pentagons*** can be considered, the most relevant is the non regular polygon achieved by means of two right triangles over the equal sides of the Cordovan triangle, (Figure 6). This pentagon covers the plane. It has four equal sides of c/√2 and the remaining side is 1. Its angles are 5π/8, π/2, 3π/4, π/2 and 5π/8.

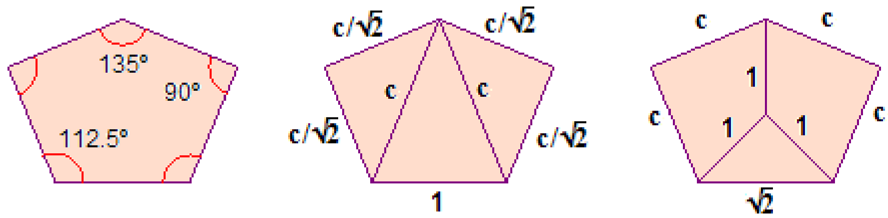

Fig. 6: A Cordovan pentagon.

The gallery of Cordovan polygons is wide (Redondo and Reyes 2009). Indeed, the collection of par-polygons is especially interesting. In fact, combinations of two kites, two darts or one kite and a dart produce different par-hexagons, (see Figure 7).

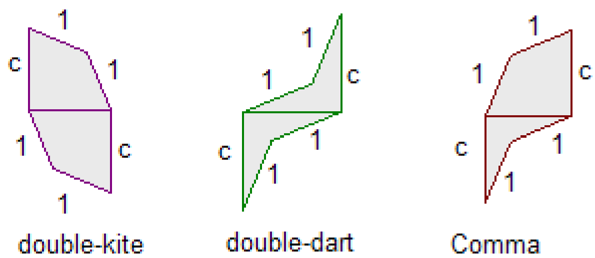

Fig. 7: Cordovan par-hexagons

In a similar way, by considering four quadrilaterals, kites or darts, several par-octagons can be obtained: the "***c-sun***" (regular octagon), the four point ***"star"*** or (***c-star***), the ***c*-bow,** the ***c-umbrella***, the ***c-moon***, the ***c-fish***, etc. See Figure 8.

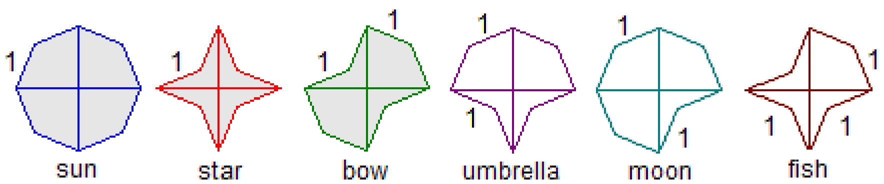

Fig. 8: Several par-octagons made with kites and darts

2. Hashim Cabrera: the proportion of the soul

Hashim Cabrera was born in Sevilla in 1954, soon afterwards he moved to Almodovar del Rio (Córdoba, Spain) where he now lives. He is a painter and writer interested in both visual arts and its historic, spiritual and philosophical basis. After a first stage, where the artist was concerned about the form, he evolved towards a *naturalist abstract* concept of the painting. By avoiding shapes and symbols, the artist's message can be freed of its servitude to the form. Under these assumptions, *colour* and *proportion* are the only elements allowed.

In Córdoba, in February 2008, he exhibits "***Los colores del alma***" (*Colours of the soul*), ten paintings (acrylic/canvases) which are a sample of his profound reflection about colour. All works are conceived as the union of two, three or four rectangular modules; each of them in a single colour. The complete exposition can be found in http://www.hashimcabrera.com/galeria.html.

Cabrera claims "*since we can't live an absolute oneness permanently, then oneness and diversity can't be separate*". This idea is represented by the main use of green, the only primary colour which at the same time is secondary. The harmony is achieved by means of a balance between the pieces. The manner of the division of the support space is purposely arranged by the Cordovan proportion. In addition, all compositions are also interrelated by means the Cordovan proportion.
Surprisingly, the Cordovan number appears to be involved in the measured values for the annual mean sunlight in Cordoba. This striking fact attracts attention of the painter, who is fascinated by that coincidence which joins an external phenomenon, macrocosmic, with an inner experience based in the mathematical concepts which arrange the order and consensus. That is, the Cordovan proportion.

We have checked and we have found this ratio in the dissections of the canvases the Cabrera's exposition, even in those cases where it is not evident. As a sample, we will focus in the paintings showed in Figure 9. From these, some geometric facts are found.

a) Díptico verde cromo/negro (diptych chrome green /black)
b) Visitantes (Visitors)

c) Tríptico verde/negro (Triptych green/black) d) Vergel (Fruit garden)

Fig. 9: Four canvases of Hashim Cabrera (Photographs by Bruno Rascado)

The picture in Figure 9, *a*) is an almost perfect square, formed by the union of two modules which shape the simplest Cordovan dissection of a square:

"A square may be divided into a Cordovan rectangle and a rectangle of ratio c/(c–1)"

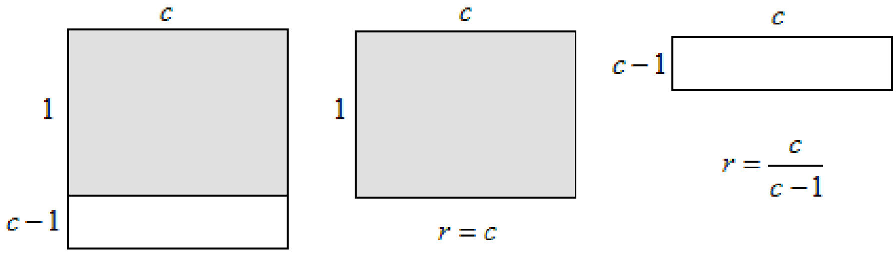

Fig. 10: A square is the union of a Cordovan rectangle and a rectangle *c*/(*c*–1).

The graphic analysis of the canvas is showed in Figure 11. On the left, we can see the painting. In the centre we observe the aforementioned dissection of the square, made with accuracy by means of a geometric design program. On the right, through a simple superposition of the preceding images it is shown the "almost perfect" Cordovan dissection achieved by Cabrera.

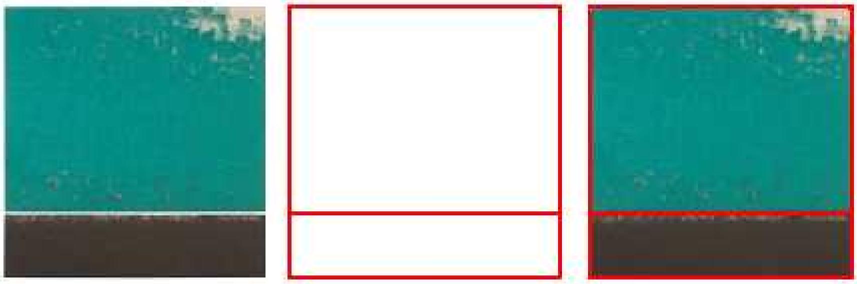

Fig. 11: Cordovan division on *Diptych chrome green /black*.

Starting with the dissection of Figure 10, we can move the rectangle to place it in the centre of the square. The result is another Cordovan dissection of the square, Figure 12:

"A square can be divided into one rectangle of ratio c/(c–1) and two rectangles with ratio 2c"

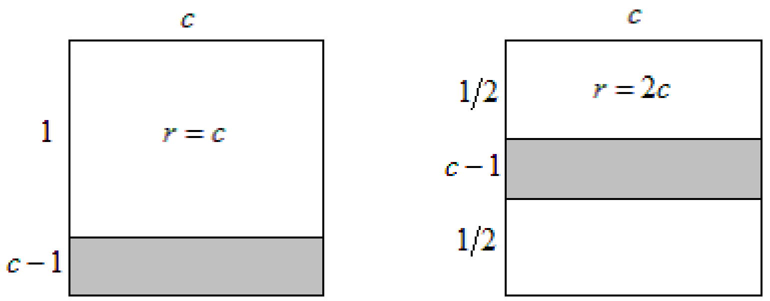

Figure 12: Dividing the square into rectangles with ratio 2*c* and *c*/(*c*–1).

We have recognized this dissection in "Visitors", where we observe a white central strip dividing symmetrically the canvas. Precisely, that strip is a rectangle with ratio *c*/(*c*–1).

Figure 13 explains without words the graphic analysis of this painting showed in Figure 9, *b*).

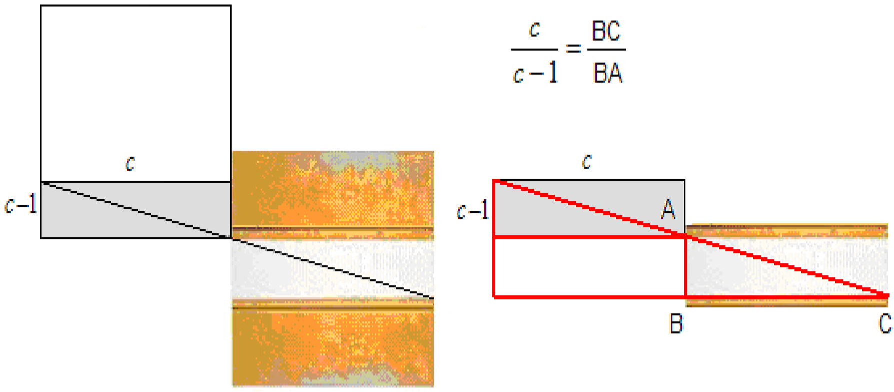

Fig. 13: Cordovan division on *Visitors*.

Starting again by the first division of the square, we can achieve an harmonic dissection of a rectangle of ratio $c/(2c-2)$, Figure 14.

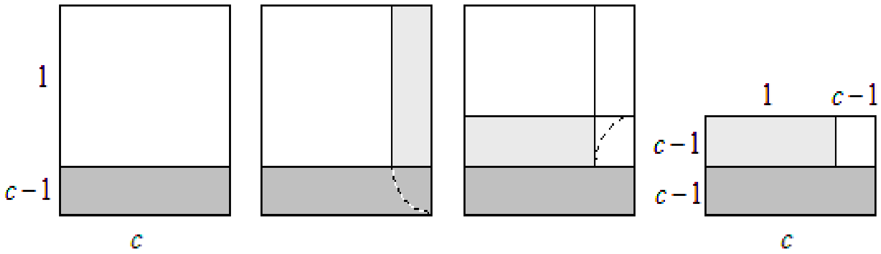

Figure 14: Construction and dissection of the rectangle $c/(2c-2)$.

This pattern appears in Triptych green/black, Figure 9, *c*). This assertion is explained in the Figure 15. Observe that both blue lines are parallel and the upper line is the extension of the diagonal of the white rectangle $c/(c-1)$.

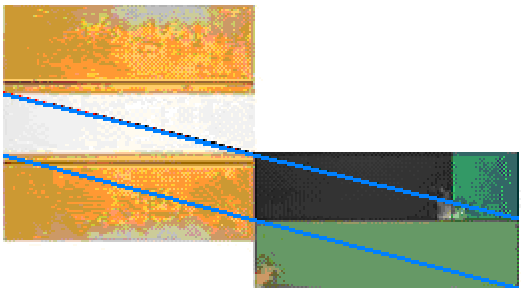

Fig. 15: A rectangle $c/(2c-2)$ in *Triptych green/black*.

Figure 9, *d)* shows *Fruit garden*, this composition is surprising. The painting is formed by a squared module adjacent to another rectangular one. It is the only case where the artist recognized that the proportions in this composition are considered by chance. It is striking that even in this case the Cordovan proportion turns up. See Figure 16.

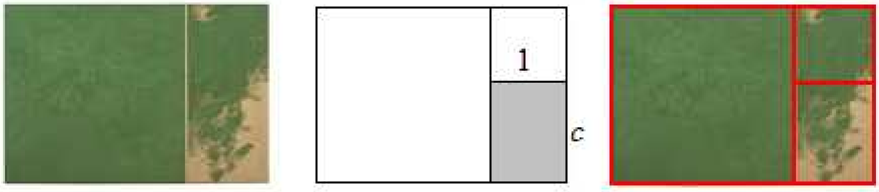

Fig. 16: Garden/Orchard towards the Golden rectangle.

Indeed, the rectangular module is formed by the union of one square and one Cordovan rectangle.
We discover the two first steps of a pseudo-gnomonical growth associated to a Fibonacci sequence $G_n = G_{n-2} + G_{n-1}$, $n = 2, 3, 4,\ldots$ with initial terms $G_0 = c$ and $G_1 = 1$:

$$c, 1, c+1, c+2, 2c+3, 3c+5, \ldots$$

Observe that

$$\frac{G_{n+1}}{G_n} \to \phi \ \ (\textit{Golden mean}) \ \text{ and } \ G_n = F_{n-1}c + F_n, \ \ n = 1, 2, 3, \ldots$$

where F_n is the term of the usual Fibonacci sequence $F_n = F_{n-2} + F_{n-1}$, $n = 2, 3, 4,\ldots$ which starts in $F_0 = 0$ and $F_1 = 1$. So the intuitive shape by the painter may be considered as an approximation of the Golden rectangle.

3. Luis Calvo: A vectorial drawing of the mosque of Cordóba

Luis Calvo (1959-) is a designer, an ardent admirer of the history and architecture of Córdoba, his home town. In December 2008, the Cervantes Institute in Brussels organized an art exhibition devoted to intra-cultural expressions of Córdoba city, through the work of eight artists. Calvo contributed to the collective exhibition with the painting, “La Proporción Cordobesa” (The Cordovan proportion), Figure 17, where the author gives homage to the Mosque, the most emblematic building of the city. Calvo’s work is a vectorial drawing on cotton fabric, on frames which are arranged forming a rectangle, which represent the plan of the Mosque of Cordova.

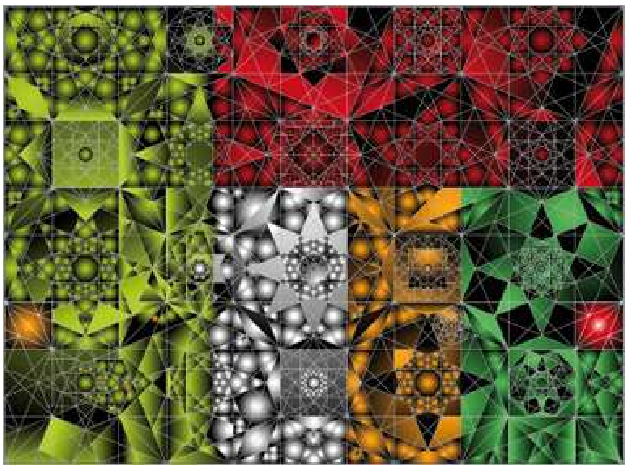

Fig. 17: La Proporción Cordobesa (2008)

Looking at the picture, we can see five rectangular modules, each of them coloured, differently in accordance to the different stages of the construction of the building. On the full rectangle, we observe many octagonal stars. Each one is inserted in a geometric framework formed by a rectangular grid jointly with another grid which has been rotated with respect to the first one.

Let us focus our attention on the geometric skeleton where the octagonal stars are placed. Its mathematical interest becomes evident when we construct it.

Dissecting a regular octagon inscribed in a square, using lines that are parallel to the sides of the square, Figure 18, we may determine a compound figure formed by four congruent rectangles of ratio $\sqrt{2}$, four congruent squares and a big central square. This module will be the basis of the initial grid.

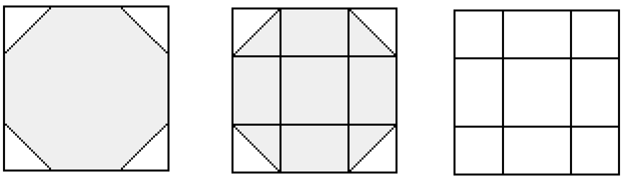

Fig. 18: Basis of the initial grid

The first figure to be constructed is the star octagon, denoted by 8/2, (Figure 19 *a*), which is formed by two squares. In the next step, inside the previous star, we draw the star octagon 8/3, or octagram, with its points lying in the convex vertices of the star 8/2, (Figure 19 *b*).

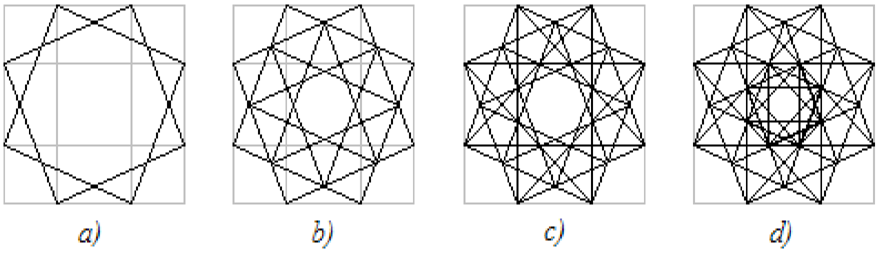

Fig. 19: Generating the basis star

Returning to the first star 8/2, we draw another star 8/3 with its points in the points of the initial star 8/2. This operation determines eight interlaced Cordovan triangles with its angles of 45 degrees in the corners of the stars, (Figure 19 *c*). Within the little internal star 8/2 determined in the centre of the composition, we repeat the same operations described in the previous lines. This composition is translated over the grid in two independent directions, Figure 20 (left). Finally, we install another identical design over the inner canonical module surrounded by the four previous modules. The final result is a grid of five interlaced modules, Figure 20 (right).

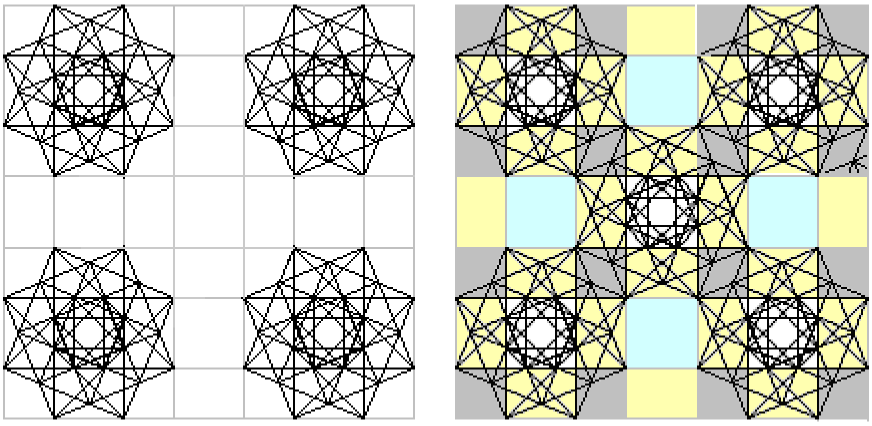

Fig. 20: Generating the pattern basis

By means of two independent vectors, we can translate the five compositions in order to achieve the picture of a tiling of rectangles and squares, some of them decorated by segments involved in the octagon geometry, Figure 21.

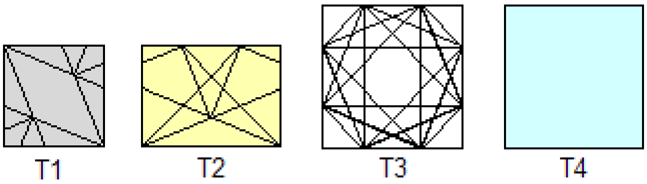

Fig. 21: Rectangular and squared modules in the grid

It is important to remark that the artist has used a very different construction procedure. The point of start of Calvo is the strategic placement of the T1 module, in the small squares of a rectangular grid as in Figure 18. Next, the segments are extended and the drawing is completed by adding the remaining lines. See Figure 22.

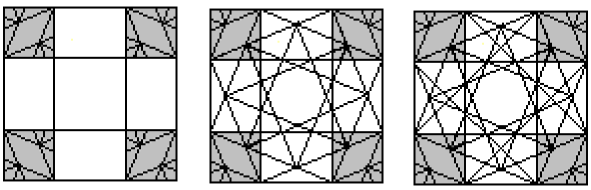

Fig. 22: Construction from T1

In the T1 module, the Cordovan diamond appears purely inscribed in a square constructed by bisecting the angles of 90 and 45 degrees (see Figure 4). Let us observe that this shape is one of the two bases of origami: *the Fish base*. In the 8th century, the art of paper folding was very popular among the Arab people. The personal proposal of Calvo is that the Mosque of Córdoba was constructed by master bricklayers who knew and used origami techniques. This is the reason for the main role of the module T1 in the way followed by the artist.

Through both preceding alternative constructions, the final result is the same, a modern "tastir" (straight line geometry). The Cordovan proportion emerges from this gallery of stars, where several Cordovan polygons are easy recognizable, which have not been drawn consciously, but appear in an implicit way. We discovered in the grid some new polygons which we could be added to our previous collection.

In order to clarify the understanding of the canvas, we are going to consider it as an array of eight rows and eleven columns, and we will codify each one of the boxes or set of boxes, as explained in Figure 23. The background of this Figure is a previous sketch of the painting kindly provided by the author.

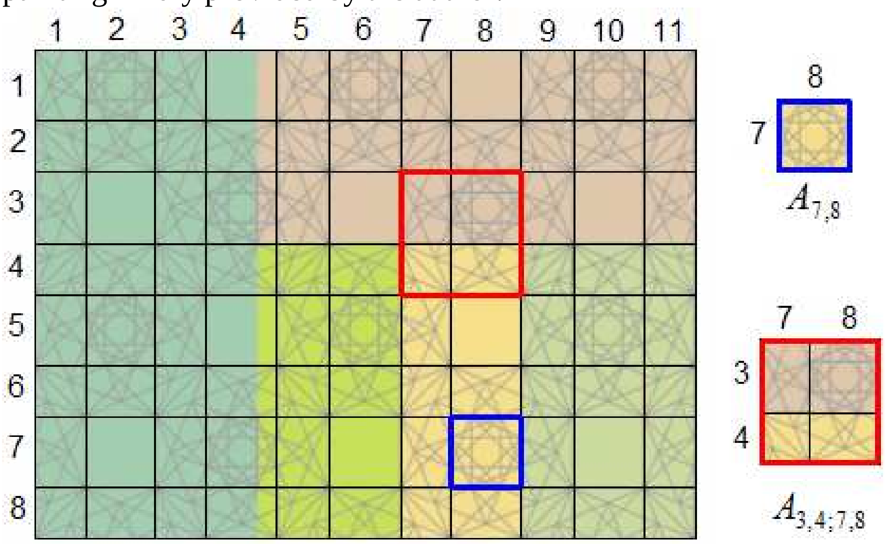

Fig.23: Codifying the canvas.

The Cordovan diamond is found off in several boxes. For example in $A_{i,j}$, with i=2, 4, 6, 8 and j=1, 3, 5, 7, 9, 11. The Cordovan triangle is spread around the picture. For instance, in the box $A_{4,5;2,3}$, we recognize two Cordovan triangles, respectively divided by each other and its respective gnomon. A *c-bow* is located in $A_{3,4;3,4}$. A concave octagon, known as *star* and a *c-umbrella* are located explicitly in $A_{3,4,5;2,3,4}$. There are octagons *c-fish* and *c-moon* in $A_{5,6,7;4,5,6}$, and so on, Figure 24.

By analyzing the picture, we can define about some new pentagons which will complete the family of pentagons discovered by the authors. These new Cordovan pentagons are cut into rectangles, squares, cordovan triangles and gnomons, see Figure 25. The first pentagons involves the Cordovan number and the Silver number $\theta = 1+\sqrt{2}$. They appears in the rows one and two passing for several columns: $A_{1,2;1,2,3}$, $A_{1,2;5,6,7}$, $A_{1,2;9,10,11}$. Symmetric pentagons of these ones can be found in $A_{4,5;1,2,3}$, $A_{4,5;5,6,7}$, , $A_{4,5;9,10,11}$.

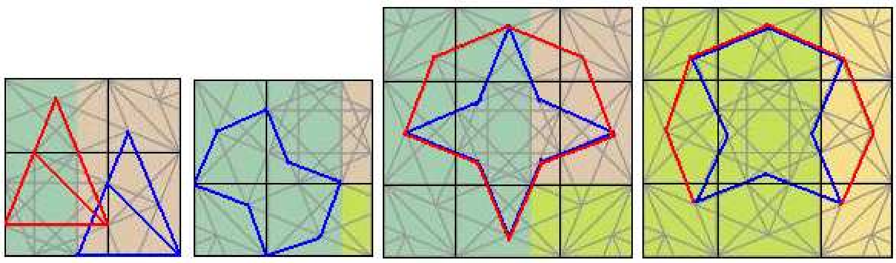

Fig. 24: Cordovan polygons in boxes $A_{4,5;2,3}$, $A_{3,4;3,4}$, $A_{3,4,5;2,3,4}$ and $A_{5,6,7;4,5,6}$.

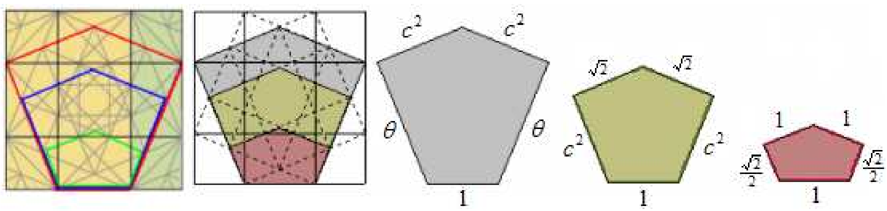

Fig. 25: Three pentagons in the box $A_{7,8,9;6,7,8}$

Looking at the T2 module, we discover several geometric facts about the rectangle √2. The upper row of Figure 26 explains the steps of construction of T2 module. In the first picture of the procedure becomes evident that "*a rectangle √2 can be divided into a Silver rectangle and two rectangles of ratio* c^2".

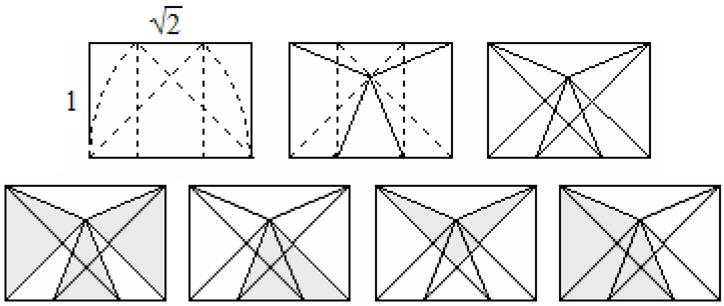

Fig. 26: Dissections of the rectangles √2

In the bottom row, on the left, we can see another dissection of the rectangle √2. In this case the result is the union of three Cordovan triangles, an isosceles triangle and two symmetric scalene triangles. That isosceles triangle is the gnomon of the scalene triangle, which is precisely the gnomon of the Cordovan triangle. The remaining pictures show the location of a *c*-dart and a *c*-kite in the rectangle √2.

Conclusions

We can conclude that the Cordovan Proportion is not just a mathematical invention. It is not a simple matter to coin a new geometric term, but it is considered by artists like Cabrera and Calvo in their art works. Research in two very different fields, mathematicians and artists converge to a common point: The Cordovan Proportion, the ratio which emerges from the soul of the Mosque of Córdoba.

Acknowledgements

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