MATHEMATICS AND ART III

Second ESMA Conference

September 18 – 20, 2013

University of Cagliari, Cagliari, Italy

http://www.math-art.eu

Conference venue

Aula Magna Matematica, Palazzo delle Scienze, via Ospedale 72, Cagliari

Exhibition venue

Cittadella dei Musei, Piazza Arsenale 8, 09124 Cagliari

Main themes of the Conference

Mathematical tools and software for the creation of artistic scientific visualizations
Analysis of artistic works from the mathematical point of view
Pedagogical uses of scientific artistic works.

Invited Speakers

François Apéry (F)
Claude Bruter (F)
Francesco De Comite (F)
Renato Colucci (Co)
Vincenzo Iorfrida (I)
Dmitri Kozlov (R)
Richard Denner (F)
Douglas Dunham (USA)
Michele Emmer (I)
Massimo Ferri (I)
Gregorio Franzoni (I)
Livia Giacardi (I)
George Hart (USA)
Patrice Jenner (F)
Marcella G. Lorenzi (I)
Andreas D. Matt (D)
Francesca Mereu (E)
Alexei Sossinsky (R)
Daniela Velichova (SK)

Organizers

European Society for Mathematics and the Arts (ESMA)

Local Committee

G. Bande, R. Caddeo, B. Cappelletti-Montano,
G. Franzoni, A. Loi, S. Montaldo, M. Musio, P. Piu

Scientific Committee

F. Apéry, C. Bruter, R. Caddeo, A. Fomenko,
M. Field, D. Kozlov, A. D. Matt, R. Palais

Sponsors

Università degli Studi di Cagliari, Dipartimento di Matematica e Informatica, GNSAGA – INdAM,
Piano Lauree Scientifiche, Assessorato alla Cultura Cagliari, Cantina Sociale Santadi

Claude P. Bruter *Editor*

MATHEMATICS AND ART III

Visual Art and Diffusion of Mathematics

CASSINI

Editor

Claude Paul Bruter

ESMA
Institut Henri Poincaré
Paris, France

Du même auteur:

Mathematics and Art: mathematical visualization in art and education, Springer-Verlag 2002, ISBN 3-540-43422-4

Mathematics and Modern Art, Springer-Verlag 2012, ISSN 2190-5614, ISBN 978-3-642-24496-4

ISBN 978-2-84225-188-8

Preface

The second Conference of the European Society for Mathematics and the Arts (ESMA) was held at the Department of Mathematics of the University of Cagliari (Italy-Sardegna) from 18 September to 20 September 2013. This volume gathers together the texts of the majority of talks held during the conference.

Since 2010, when the first conference was held in Paris, minds have been considerably and recently rapidly evolving. It seems now widely and quite accepted that the marriage of Art with Mathematics has a positive feedback on the intelligence and the acceptance of mathematics by most of the public.

When planing the conference, three main themes were defined:

Theme 1: Mathematical tools and software for the creation of artistic scientific visualizations

Theme 2: Analysis of artistic works from the mathematical point of view

Theme 3: Pedagogical uses of scientific artistic works

The main scope of the lectures was to give useful old or new tools to a wide audience to get in touch with some mathematics that artists could use to create new works. By focusing the attention on these tools, the idea was not only to enlarge the possibilities of creation by artists, but also to invite mathematicians to maybe locally deepen some of their theories. An other aim was the promotion of math and art activities in the educational system, which are going on now. There is only one contribution entirely focused on this third theme, emphasizing the role of exhibitions associated with lectures, but most other contributions, concerning models, have some connection with it.

The historical article by Livia Giacardi mainly refers to the works of the Italian school at the end of the 19th century and the beginning of the 20th. At that time, algebra and analysis had not taken the important role they have got today, and algebraic, analytical and differential geometry played an important place in the teaching at any level. Some models and works are described and analyzed in the articles by most of the speakers. The beautiful models in wood made by Joseph Caron during the same period are now currently exhibited in the library of the Poincaré Institute. They have been mathematically reconstructed by François Apéry. In some sense, Joseph Caron was a pioneer of singularity theory. This kind of work may be understood as an encouragement to study generalizations of physical and mechanical devices to higher dimensions giving rise to richer mathematical configurations and illustrations. A question remains without answer: for which courses did the Italian geometers and Caron use these models, and what was the pedagogical success of that uses ? Note that the series of polyhedral models constructed by Richard Denner to illustrate the sphere eversion is typically in the spirit of all the previous models since they intend to be used as pedagogical tools. As George Hart did, he gives recipes to build the models.

But George' motivations are different from Richard's in the sense that his artistic passion was the main incentive to create his well known, beautiful, attractive and definitively original polyhedra. He gives the very detailed clues to reproduce them with a large size, and relates the various successful pedagogical experiences he made in the American context, where large groups of students gather to build up the artwork.

There is some similarity between that presentation and Dmitri Kozlov's one. Dmitri shows several people working at reconstructing one of his kinetic modules. These modules are cyclic periodic knots he first carefully studied as mathematical objects in a previous work. Materialized in metal or in fiberglass, being able to be constructed at any size, they can be used in architecture and for education as well. One of their characteristics is flexibility both in the material sense and in the conceptual sense. They can be deformed, so that the same knot can move into the frame of a sphere, of a torus or of an hyperboloid. From the pedagogical point of view, these original modules share a new mechanical and mathematical interest. Some analysis not only bring some new highlights on the works, but also allow to improve and extend them. That is for instance the case of Dough Dunham' study of Escher's work in 2-dimension hyperbolic geometry from which he constructs aesthetic triply periodic polyhedra. We are entering again the domain of real artistic works arising from the mathematical world.

Using various mathematical theories among which dynamical systems and time series analysis, the refined analysis of the scores of some well known composers by Renato Colucci and his colleagues allow them to propose algorithms which approximatively simulate the content and the structure of these scores. Then they can play to create interesting pieces of music which mix up different composers.

The work by Francesco De Comite is not focused on new mathematical research giving birth to new mathematical objects, but on a clever artistic use of former mathematics through recent computer tools he masters. He is a well known graphic artist, and shows a few of his original works coming from his favorite mathematics. Though he comes from computer science, one can say that, being a specialist of anamorphosis, he has a topological mind in the sense that he likes deformations, using the flexibility given by parameters. Bifurcation theory is related to creativity and life.

Artists or not, most people are today familiar with standard symmetry with respect to mirrors, or affine linear subspaces. There is no standard symmetry without underlying stability. On the other hand, standard symmetry is a special case of a more general symmetry with respect to any manifold, and which has to be seen first as local. When the manifold is a n-dimensional circle, the analytical formulation of that symmetry is called inversion. The article by Renzo Caddeo, Gregorio Franzoni and Paola Piu uses this inversion to fold a beautiful surface named the Dini surface, and to create a nice bouquet of such Dini surfaces. Nowadays, with the weakening of the teaching of geometry, inversion might be unfortunately ignored by many students. They will find here the possibility to get familiar with it, while artists could use it to get nice deformations of some of their objects and to improve their creations of works.

From the fact proved by Nash and Tognoli that any C^∞ manifold has a real algebraic model, polynomial representations are the most frequent to appear in the mathematical literature devoted to geometry and topology. I would like to emphasize the usefulness of the Minkowski tricks used by Daniela Velichova to construct new shapes and which can be used to create numerical and visual representations of new mathematical objects that artists could use. I would like also to mention the fact that, apparently, there is no paper devoted to the mathematical study of this kind of representation.

I suspect that general cones defined by the last author could be better visualized through the previous means. I shall not comment anymore this theory, the introduction and the conclusion of the article are quite explicit.

During the Conference, we could visit a nice mathematical exhibition inside the "Citadelle dei Musei", and applaude a joyful play of theater written by our hosts. All the participants would like to thank Renzo Caddeo and his colleagues for the perfect organization of this Conference, in a warm atmosphere.

It is a pleasure to thank Catherine and André who welcomed us at Cassini. This publication is made possible through a grant from Cap'maths as part of the Investissements d'avenir (Future Investments) program.

Claude Paul Bruter

Contents

THE PEDAGOGICAL VIRTUES OF MATH AND ART EXHIBITIONS

Claude P. Bruter

Abstract

This article deals with exhibitions of artistic mathematical works as useful tools to fight against the disaffection of the public towards mathematics.

1 Problems of today and of yesterday

The subject which we will treat in this article concerns disaffection of various publics towards the sciences, and more particularly towards mathematics. How to succeed in reconciling these publics with mathematics in particular, the role of which is fundamental in the understanding of our universe, of our environment, in the conception, the creation and the implementation of all these means among others technical which facilitate our lives and which prepare our future? What are the contents of the messages which we should send, be assimilated by these publics, and how can we achieve this?

Many of us share the conviction that artistic work can be a useful tool for the diffusion of mathematics, their initiation, their teaching. But it is a fact that many of our fellow-citizens, of our decision makers, even some of our colleagues, are not convinced of the merits of that approach, either due to lack of judgment and reflection, or quite simply from ignorance.

This article intends to address those not very familiar with mathematics and arts decision makers, and those colleagues who are not totally convinced. Being mainly focused on exhibitions, it is a very partial introduction to the subject.

2 The foundations of the argumentation

It is obviously impossible to handle in detail and in a short article the multiple aspects of the elementary questions which have just been introduced. I shall have to content myself with making here one excessively short and synthetic presentation of some points of view and partial solutions that I have already had the opportunity to expose and to experiment with.

The following arguments constitute their common foundation:

an argument drawn from general biology: in all the aspects of development and evolution, ontogeny recapitulates more or less, and is a reflection of phylogenesis, in particular its important accidents. As a consequence, in order to teach and to lead to a better understanding of the most modern theories, it is necessary to carry out an intellectual journey analogous to that which has accompanied, over time, the progressive implementation of the basic notions and the essential facts, since their most distant origins.

an argument related to our physiology: connected to the general vitality of the body and its driving system, the primary sensitivity is that of the senses, enabling us to immediately comprehend our world, its dangers and its advantages. Very young children, up to four-five years of age, have a global vision of space, a spontaneous prehension of 3D. The view fixed to the plane, the mental exercise, practiced during school years, the young years of maturation of the brain, and restricted to what takes place in two dimensional space, tend to make the imaginative vision in the three dimensional space difficult later.

an argument also related to our affectivity: the most of us assign to the objects that we meet aesthetic qualities to which we are often sensitive to varying degrees. Our strongest and most long-lasting impressions are generally those which have struck our emotions.

These primary concepts are strongly present in the first part of a work entitled *Comprendre les Mathématiques* [1], about which the late Gustave Choquet, announcing his "admiration for this beautiful success", wrote, this "book merits to be read and reread".

These are also the underlying concepts of the ARPAM project, in the form of exhibitions of works, conceived from mainly mathematical considerations, remarkable enough to be able to be qualified as works of art.

The value of a work, whatever it is, is measured by the quality of its realization, the originality of its conception and by its contents. It is the unexpected which strikes and which attracts, here is for the appearance, but in the end, we consider its significance and its contribution to universal well-being, knowledge and intelligence.

It is not always immediately obvious to appreciate this value, to recognize it and to comprehend it in all its aspects. The guidance of a third person is never useless to reveal these contents.

3 Some critical observations

Do we flee from the mathematics? Why? Most likely because what we teach is hardly apparent in the physical world and is not physically perceived, has no immediate meaning for the body, does not awaken the senses, requires an effort for the spirit to give it any tangible reality. The effort is mostly without profit, disappointing, disheartening for some. What we show is fragmented, without reference to history which shows the justified genesis of things, thereby confirm their presence. We do not understand where we are being directed, or how this present joins in an organized ensemble, reassuringly solid.

Its embodiment in the material object gives to the mathematical abstracted object a physical reality, the image and functional character of which can be all the more comprehended and felt by the spectator. If this functional character is absent, only particular properties of this material object will allow its image to join the memory, to assimilate to the cleanly vegetative sense of the term the essential features.

It will be the case in particular if it presents asserted aesthetic characters.

These are thus the principal reasons for which the mathematical objects will be accepted, will have their recognized presence whether it is in the form of models and small sculptures or representative drawings emphasized by the genius of the artists. We shall no longer, we shall wonder.

In the light of these data and of these common sense facts, it should come as no surprise that the young generations show a lack of interest in mathematics. The current trend, under the pressure of some professional mathematicians, physicists and engineers, is to favor calculus to the detriment of geometry, while to the young children, the number is almost meaningless, unlike that of the figure which has a pregnant physical and affective meaning. In the current teachings of many countries, the number, the letter and their use take it on any other consideration. In the mind of a child what do these two attached symbols 11, and the number 11 mean: a friend, the cat, a car, a bar (of chocolate)? And $x^2 + b = 0$ or $\sqrt{2}$: that is drinkable, that is edible, we can throw it then catch it? It is through the exercise, mental calculation, the acquisition of the multiplication tables that the child can become familiar with number, even if the numbers have no significance for him. On the other hand, the drawings of a triangle, a polygon are physical structures which make sense, to be associated with the shape of common objects.

Thus rather than to eliminate it, it is on the contrary the “monstration” of shape and its properties that has to be the object first and foremost of all the attentions in the early education of mathematics. The simple drawing of a circle is enough to immediately show one of its beautiful properties, the most wonderful doubtless, the one that surprisingly the circle, the only one among the infinity of flat shapes, has the privilege to possess. The role of the mediator here will be to help the creator of the circle, the attentive spectator, to make the formulation of its observation mature and to help him to express even this formulation in exact terms, to give birth would have said Plato to what is being developing in his mind.

4 The charms of the exhibitions

These exhibits allow the public first to discover the richness of the mathematical world, and second to allow that public to get acquainted with some quite modern mathematics, completely different from what they learned at school, and without any mental stress. The exhibits are thus by no means boring, they are felt beautiful and amazing, intriguing, enriching, but of course they may leave a feeling of dissatisfaction from the fact that the meaning of many of the works is not quite understood, the feeling that a large gap remains between the visitor and the content of the works he has been admiring. If the psychological position of most of the visitors towards mathematics remains somehow ambiguous, in any case, the psychological resistance against mathematics has diminished, and this is a true first success.

The exhibitions of models, small sculptures and printed works, whatever the medium may be, show mathematical objects of generally recent conception and discovery, emphasized by illustrators and quality artists.

These are main advantages in their favor. They present a wealth and beauty that are often missing in the oldest objects, and it is their very novelty which attracts the curious and the crowds. Has not the term "fractal" become a household word for example?

That they were the object of recent attentions on behalf of the mathematicians, make us think that it is the most current mathematical theories which were of use to their discovery and to their study. In other words that in the presence of these objects, we are in fact at the heart of modernity. And if thus we bring to the public some at the same time simple but penetrating explanations on their subject, the same auditors will be pleased to feel in sync with the most astute current events. Why then would they reject mathematics? They are beautiful, multi-form, and thought well in their accessible foundations.

Models and small sculptures have the advantage over the printed works for they can be touched, manipulated and examined from every angle. Some of them can be knocked down and built up again as one pleases, adding to their charm for the handymen. They then become occasional pastimes.

The most interesting of these objects are doubtless the ones which are realized with threads. We can enter inside objects, allowing us to view hidden aspects in their structure.

And among these objects in thread, are the deformable ones which illustrate additional properties. They can take unexpected forms, surprising dynamic behavior. Children in particular love to manipulate these objects, which have now an appreciated playful side.

Finally, rigid or not, suitably lit, their shadows on well chosen surfaces add to their charm and to their interest, so much rich is the mathematics of the visible outlines.

I do not doubt, in this place of my presentation, that auditors and readers are perfectly aware of the value of the exhibitions to contribute in an effective way to easing the fears felt by a lot of public towards mathematics, and to lead to a positive vision of our scientific universe, through the enhanced appreciation of the hidden beauties of mathematics.

5 Taking Advantage of the Pedagogical Content of the Exhibits

If these exhibitions and their contents constitute a quality media tool, it would be disappointing if we could not use them also for more advanced educational purposes. Indeed, it would be a shame if the entire content of our collection, with a large pedagogical potential, remained asleep in some dark dormitories.

We can only regret that the so-called popular annual neighborhood events–where we see some classes of small merry pupils stopping for one moment with their guide in front of tables with so-called educational, books, games and various objects–of so little use to the training of the mind. What do these day visitors learn of mathematics? We show polyhedrons, we make some simple tessellations. What could the spectator glancing at the icosi-thingumajig learn? What new knowledge of mathematics did he come away with his superficial visit?

The exhibitions, of course, do not escape the same criticism. It is then advisable to value the contents. We reach the public by presentations made in a relaxed atmosphere, in the unusual setting of an enlightened public in a setting containing some of the beautiful works which will be commented on.

The content of the presentation obviously depends on the public being addressed. Age, educational level, and audience reactions are important factors. There is no ready-made formula.

Of course the way the presentation is made is important. It should by no means be an academic presentation. It is better to be joyful and enthusiatic. The speaker communicates his or her surprise at a property or particular fact, showing philosophic seriousness in front of such or such property or general, universal fact, coming and going from an example to the other one in apparently different situations which illustrate this fact, creating the global vision of a theory within which the same fact joins, evoking the history and how it is connected with other theories. Let us be desire that the auditor leaves the room relaxed, but also with the understanding of a new general concept, the knowledge of a new particular fact. Then maybe he will have the impression to have reached the threshold of the prestigious universe of mathematics, to have crossed a door and made a first step, the very first step, humble and reserved, in this world which formerly frightened him, and which seems to him today simple and radiant. Hopefully he will no longer fear mathematics, and will share this feeeling with others.

A large audience will accept and be interested in the main facts, in the main ideas and concepts behind the works, among which, above all, the concept of stability which has not yet been quite well understood. The presentation of these concepts should include a few words about their history, the extent of their incarnation, their importance to the physical, mathematical, artistic and philosophical worlds, and when possible some easy and fast explanations as well. Given in a favorable environment, these talks give rise to exchanges between the enthusiastic speaker and the audience, in a relaxed and rather joyful atmosphere.

The contents of these presentations at which it is hinted here will very likely arouse some reserves on behalf of the mathematicians. For the professionals, a good mathematician is the one who shows his community new properties, and who explains the reasons for their presence, who gives the proofs. The value of a researcher lies in creating new concepts and theories, in discovering new properties.

To see properties demands a familiarity and an attention which the public cannot acquire alone and be self-sufficient in three quarters of an hour. However, a speaker can very well during this short lapse of time succeed in awakening the attention of his public, and in guiding it to understand significant properties of the objects which it could not distinguish at first sight. So low it is, the presentation so conceived in the presence of works of art possesses an educational quality which we cannot underestimate. Still it is necessary to have warned the public beforehand about this fundamental aspect of the work of the mathematician.

We can only wonder that it was necessary to wait for year 1742 to notice that any even number is the sum of two prime numbers. Is it possible that the great Greek mathematicians had not realized this elementary fact? It happens that we pass next to the beauty which lines our way without becoming aware of it, and that the simplest things also are the most difficult to understand and to justify (prove). The mathematician who is able to prove the greater part of Goldbach's observation is destined for fame.

The second quality of the mathematician is to know how to give the irrefutable proof of the existence of a property. A certain dose of trick, but also the familiarity with the various already used technics of demonstration, their control, the knowledge of numerous already well established properties are necessary to well lead new demonstrations. It is the reason why the professionals grant a large importance for any kind of exercise which develops the mental agility. And besides, an understanding of the demonstrations reveals a thorough knowledge of objects, generally facilitating the discovery of up to now unobserved properties.

The presentations address publics which can be qualified as virgin in mathematical subjects for the most part. Is it then reasonable to make detailed presentations before them? In these conditions, is the presentation, seen from the point of view of the professional mathematician, interesting enough to encourage a more widespread audience?

These last comments deserve to be qualified, because they are the expression of a somewhat extremist position. We know well that except for some very rare specialists, to go into detail the knowledge of the complete proofs of the ex-conjectures by Fermat or Poincaré is a matter of illusion. Happy already the one who can know the main lines of these demonstrations, the reasons which led to choose such or such mathematical tool, some of the progress which uncorked in the obtaining of results appearing now particular, intermediaries, but interesting in themselves. In front of the scale of certain demonstrations, the mathematician eventually eventually is satisfied with the knowledge and with the understanding brought by the datum of these guidelines.

During the maths and arts lectures, it can occur, in some rare and simple cases, and in front of certain public, that we can justify the assertion by the reasoning and the deduction, even give these indications evoked in the previous paragraph on the procedure of demonstration. But more commonly and more modestly within the framework of these presentations, it will be possible to mention the theories which come into play, their history and the subject of which they take care, their objectives, and especially to refer to physical, natural facts, these theories of which, through their statements, develop the representation and their effects.

After these presentations, the audience will therefore perhaps enrich their vocabulary with some mathematical terms denoting particular objects they have seen, or which have touched their eyes through beautiful images. These objects will have shown them some features and unexpected properties, objects of which they have in theory registered some fundamental concepts, objects inserted in theories of which the audience would have learned some fundamental concepts.

Would this optimistic comment be that one of a dreamer, even a humorist? We might think that at first, but doubt it in the light of the successful experiments made in France [2] in 2011 (Fig. 3) and in Greece [3], respectively in 2007 and

2012 (Fig. 1 et 2) with several classes of children ranging from 5 to 20 years of age.

Fig. 1

Fig. 2

Fig. 3

It would be necessary, of course, to multiply them, and especially to enrich them. Local exhibits and exposés could be set up anywhere. With the permission of the authors, and perhaps a small payment to them, ESMA could send by e-mail the images of the works that the local organizer would like to show, to use, and then to print.

One can find on our website examples of such exposés, unfortunately in French, looking at *Bonne Année* |Part I|Part II and *Pâtisserie Mathématique* |Part I|PartII|Part III|Part IV. (cf. http://www.math-art.eu/Documents/ListOfAuthors-Publications(3).php#20).

6 Conclusion

To conclude, we shall resume simply the main part of these contributions.

The exhibitions, which attract the visitors by the novelties that they can discover, by the large diversity of the works and by the beauties that they reveal, allow the public to approach the field of mathematics in a relaxed and smoothing atmosphere. Often seduced by this unexpected world, here are finally our fellow countrymen pleasantly at peace with one of the most elegant sciences.

Presentations complete the exhibitions. They bring to the visitors a first knowledge of the concepts and mathematical facts which are developed in theories. Further, they allow to appreciate the interest, the importance of the concepts which come from their relevance and the generality of their embodiment. Then placed in the basis of the theories, their genesis is intimately connected to the history of the development of mathematics. Thus these presentations, through the marked initiatory character in the field of the mathematics, where the mathematics conjugate with the arts, contribute to promote thought.

Through the open-mindedness which they bring to the intellectual world and more particularly to the contemporary scientific world, these exhibitions and presentations play a positive and original role in the insertion of the individuals within our societies.

References

[1] C. P. Bruter, *Comprendre les mathématiques*, Odile Jacob, Paris, 1996.

[2] C. P. Bruter, "L'œuvre d'art mathématique, un outil pédagogique efficace", http://www.math-art.eu/Documents/pdfs/L'oeuvre_d'art,_outil.pdf.

[3] C. P. Bruter, *Matematika* (Mathématique et Art), *A'Eukleidnz*, 85, 2012, 1-4, et 88, 2012, 1-4. Texte établi par Dimosthenis Varopoulos d'après *Pâtisserie Mathématique* |Part I|PartII|Part III|Part IV. Cf. http://www.math-art.eu/Documents/ListOfAuthors-Publications(3).php#20.

Claude Paul Bruter
Retired Professor (Université Paris 12)
ESMA, c/o Institut Henri Poincaré
11 rue P. & M. Curie
75321 Paris Cedex 05
bruter@u-pec.fr

MODELS IN MATHEMATICAL TEACHING IN ITALY (1850-1950)

Livia Giacardi

Abstract

Up to the present, the research on the collections of models in Italy has been limited to examining and cataloguing the collections existing at various universities, with a great deal of work being carried out mostly by Franco and Nicla Palladino,[0] but an in-depth study of their use in university and pre-university teaching does not yet exist. In my paper I will deal with this problem, considering the period running from the mid-nineteenth century to the early decades of the twentieth century. In particular I will focus on the following points: the lack of interest in designing and making models for teaching at the university level at the turn of the twentieth century in Italy, and its causes; the one exception to this, i.e., Beltrami's cardboard model of the pseudospherical surface; models in pre-university schools in the 1800s; Corrado Segre and the use of models at the University of Turin; the role of models in the "laboratory school" (Vailati, Marcolongo, Montessori, Emma Castelnuovo); the reconstruction of university collections of models in the 1950s. Finally I will try to draw some conclusions from this first historical analysis of the question.

1 The lack of interest in designing and making models for teaching at the university level at the turn of the twentieth century, and its causes

The use of models for research and teaching of mathematics began to spread in the second half of the nineteenth century and saw mathematicians of the highest scientific calibre engaged in this effort. The most important initiatives developed in France (Paris), in the United Kingdom (London, Manchester) and especially in Germany (Munich, Darmstadt, Karlsruhe, Göttingen)[1]. Although some first collections of models date from the early decades of the nineteenth century, the mass production begun in the seventies mainly in Munich when Felix Klein came to teach at the local Technische Hochschule and began his collaboration with

[0] See References on models in Italy.
[1] See for example N. & F. Palladino 2009 [9].

Alexander Brill[2]. In the following years various exhibitions[3] and the publication of catalogues[4] favoured the spreading of the use of models in teaching at international level. Italy remained marginal in the activity of conceiving and making geometric models, in spite of the many young mathematicians who went to Germany for post-graduate study[5]. Actually there was an attempt by Giuseppe Veronese (1854-1917) to set up a national laboratory for the production of models. In 1883 Veronese, who did post-graduate work in 1880-1881 in Leipzig where at the time Klein was teaching, in a report to the minister of education wrote:[6]

"*I believe that for increasingly substantial progress of mathematical studies in Italy, an Atelier such as that in Munich would benefit us greatly, because our schools would be made independent in this respect too from foreigners and could procure their collections at less expense*".[7]

In fact he thought that:

"*Intuition in Geometry consists in our representing in our minds the figures in space, in such a way that our thoughts can go into them, alternately uniting them and separating them, and discovering the intimate nexus that permeates them. This is the intuition of space that must be developed in the minds of young people from their most tender years, and to that end it is useful to accompany each geometric proof, as far as possible, with drawings and models by which the young person can better comprehend and intuit the geometric properties of bodies without many mental efforts*".[8]

[2] See Rowe 2013 [36].

[3] We mention the Exhibition in 1876 in London; the World Exhibition in 1885 in Anversa; the International Exhibition in 1893 in München; World's Columbian Exposition in 1893 in Chicago; the International Exhibition in Heidelberg during the Third International Congress of mathematicians in 1904; the Exhibition in 1914 in Edinburgh.

[4] *See Catalog der Modellsammlung des Mathematischen Instituts der kgl. Technischen Hochschule München, aufgestellt im Januar 1882 unter Leitung von Prof. A. Brill*, 1882 (see Fischer 1986, p. V; N.& F. Palladino 2009, pp. 51-52); *Katalog mathematischer und mathematisch-physikalischer Modelle, Apparate und Instrumente ... herausgegeben von Walther Dyck, Professor an der technischen Hochschule München, München: Wolf & Sohn, 1892, Nachtrag, in 1893; VII ed. 1903, VIII ed. 1911.*

[5] Here are the best-known of these young mathematicians: A. Tonelli did post-graduate work in Göttingen (1877), A. Capelli in Berlin (1878), S. Pincherle in Berlin (1877-1878), G. Ricci Curbastro in Munich (1877-1878), L. Bianchi in Munich with Klein (1879-1880), G. Veronese in Leipzig with Klein, (1880-1881), E. Pascal in Göttingen with Klein (1888-1889), C. Segre visited Frankfurt, Göttingen, Leipzig, Nürnberg, Munich (summer 1891), G. Fano was in Göttingen with Klein (1893-1894), A. Viterbi in Göttingen, (1897-1898), F. Enriques in Göttingen (1903), etc. In general, 71% of students preferred to carry out their post-graduate studies in Germany, with only 18% opting for France; see A. Dröscher 1992, Die Auslandsstipendien der italienischen Regierung (1861-1894), *Annali dell'Istituto storico italo-germanico in Trento*, XVIII, pp. 545- 569.

[6] The following translation and all others in this paper are mine unless otherwise noted.

[7] F. Palladino 1999 [5], p. 60. The report is reproduced in its entirety here.

[8] Ibid., pp. 59-60.

In spite of the support of important Italian mathematicians F. Brioschi, E. D'Ovidio, R. De Paolis, U. Dini and E. Bertini the initiative was unsuccessful. Starting in the 1880s, the principal Italian universities generally preferred to acquire models from abroad, mainly in Germany with reference to the catalogues of Brill and Schilling. The collections of some universities also included models from France (Muret Collection in Genoa) and England (George Cussons models of penetration in Naples). The first universities to acquire collections of models were those of Pisa, Rome, Turin, Pavia and Naples.

Initiatives for designing and producing models can be documented at the university of Naples; these however were generally limited to use within the university and there was no industrial production. In particular can be mentioned the 36 models (wood, bronze, horsehair) for the university teaching of descriptive geometry which Alfonso Del Re (1859-1921) got his students to build between 1901 and 1906[9]. In this university the use of models in teaching practice was also favoured by Ernesto Pascal (1865-1940)[10], who specialised in Göttingen between 1888-1889 with Klein and, after some years spent in Pavia, arrived at Naples in 1907, where he remained until his death. Here, as Dean of the Faculty of Sciences he reorganised the teaching of mathematics, creating for each professorship a laboratory equipped with models and instruments. In fact be believed that models were useful in both teaching and in research; it is significant that Pascal mentions models several times in his book *Repertorio di matematiche superiori* (Milan: Hoepli, 1898, 1900)[11], translated in German under the title *Repertorium der höheren Mathematik* (Leipzig, 1900, 1902). Very important was also the equipment of the Institute of rational mechanics directed from 1908 by Roberto Marcolongo (1862-1943), where geometric models in plaster or cardboard, often made by the students, were present alongside models and tools for the teaching of that discipline. From 1908 to 1934, in this Institute at least 134 degree theses were discussed, many of them accompanied by models. In his autobiography, Marcolongo writes:

"*This ample and varied material is widely used, and is of the greatest aid in the lessons of rational and advanced mechanics. There are no lessons in which, in one way or another, it is not shown to students; indeed, some of the lessons are completely dedicated to illustrating theoretical results with experiments*"[12].

Nevertheless, in Italy a systematic activity of designing and then constructing

[9] See the booklet by Del Re with the title Programma del corso e programma di esame per l'anno scolastico 1906-1907, which in Appendix includes the list of geometric models built by the students of the school of descriptive geometry of the University of Naples from 1901 to 1906. Only one of these models has survived; see F. Palladino 1992 [2], tav. 13.

[10] Pascal is well known for his integraphs for differential equations, see e.g. E. Pascal, I miei integrafi per equazioni differenziali, *Giornale di Matematiche*, (3), 51, 1913, pp. 369-375.

[11] See e.g. Pascal, *Repertorio di matematiche superiori,* op. cit. 1900, pp. 447, 469, 480, 750.

[12] Marcolongo 1935 [26], pp. 30-31.

geometric models for university teaching at industrial level never flourished. It is significant, for example, that at the International Italian Exhibition, held in London in 1888, the Education section was not awarded any prize because the publishing houses were present only with books and other kinds of publication, and these had already been evaluated in their own section[13]. Many years later in 1904 at the exhibition of models organized during the 3rd International Congress of Mathematicians (Heidelberg, 8-13 August 1904), Italy did not display anything, even though five Italian publishing houses were present[14]. The invitation to participate in that exhibition, extended to Italian mathematicians by the historian of mathematics Gino Loria in the pages of his *Bollettino di bibliografia e storia delle scienze matematiche*[15], was ignored. This situation seems to be connected to the fact that models were mainly used for educational purposes, at least according to the data known at present, and it was thus more convenient to purchase ready-made collections from abroad. Another explanation might lie in the very nature of scientific research: in the second half of the nineteenth century, in Italy three different approaches to geometric research were prevalent: the analytical approach, which was more theoretical and abstract (Ulisse Dini, Luigi Bianchi, etc.)[16]; the study of the foundations of geometry with an emphasis on logical rigour (Giuseppe Peano's School); and finally, the working method of the Italian School of algebraic geometry (Corrado Segre, Guido Castelnuovo, Federigo Enriques, etc.), whose members instead attributed great importance to intuition and visualisation. In spite of this, they did not use physical models in their research work, but preferred to employ the *Gedankenexperiment*. A famous passage by Castelnuovo describes the working method that he and Enriques used in their early years:

"*We had constructed, in the abstract sense of course, a large number of models of surfaces in our space or in higher spaces; and we had distributed these models in two displays cases. One contained the regular surfaces. At the end, the assiduous study of our models had led us to divine some properties which had to be true, with appropriate modifications, for the surfaces in both cases; we then put these properties to the test by constructing new models. If they stood up to the test, we looked for–ultimate phase–the logical justification*"[17].

An analogous reference to a sort of virtual models seems to be in the following passage in a letter of Corrado Segre to Klein:

[13] See *Esposizione Italiana di Londra* 1888, London: Waterlaw & Sons, 1888, pp. 231, 233.

[14] See *Modellaustellung*, in *Verhandlungen des dritten internationalen Mathematiker-Kongresses in Heidelberg vom 8 bis 13 August 1904*, Leipzig: G. B. Teubner, 1905, pp. 731-736.

[15] See *Bollettino di bibliografia e storia delle scienze matematiche*, a. VII, 1904, p. 64. Loria periodically provided information on the construction of new models in his *Bollettino*.

[16] The theoretical results about surfaces with constant negative curvature by Dini and Bianchi are mentioned in the *Katalog ... von Walther Dyck*, p. 292.

[17] G. Castelnuovo 1928 [14], p. 194. Cf. also Enriques 1922 [18], p. 139.

"*What you tell me about the effect that synthetic reasonings of n-dimensional geometry has on you, does not surprise me; it is only by living in Sn, (my emphasis), by thinking about it always, that you become familiar with these arguments*"[18].

This attitude of the Italian geometers towards the geometric objects that they studied is also highlighted by Oscar Zariski's review of Beniamino Segre's treatise *The Non-singular Cubic Surfaces* (1942). Zariski wrote:

"*But the reader who is willing, so to speak, to live for a while on a cubic surface (my emphasis) and to read the book in the spirit in which it has been written will be greatly rewarded by the elegance and ingenuity of the author's treatment of the subject. It is based on the principle of continuity and the method of degeneration*"(Zariski 1943).

In university textbooks of the early twentieth century as well the representation of geometric models is also rather rare and serve solely didactic purposes. The only example found up to the present in which drawings of surfaces appear to recall actual models is the treatise *Lezioni di geometria intrinseca* (Naples: Presso l'Autore-Editore, 1896) by Ernesto Cesaro, in the part regarding surfaces of constant positive or negative curvature, and surfaces with a constant mean curvature (pp. 178, 181). In university textbooks of differential geometry and projective, descriptive or analytical geometry of the beginning of the century (L. Bianchi, U. Dini, F. Enriques, F. Severi, G. Castelnuovo, A. Terracini, G. Fubini and E. Čech) figures are either not introduced, or they are simple schematic representations of conic sections, quadric and pseudospherical surfaces [19].

2 One exception: Beltrami's cardboard model of the pseudospherical surface

As illustrated above a systematic activity of creating and building geometric models for university teaching did not take roots among the Italian mathematicians. The only exception to this situation is represented by Eugenio Beltrami (1835-1900), who, as it is well known, in his *Saggio di interpretazione della geometria non euclidea* (1868) provided an interpretation of the lobačevskian plan by means of surfaces with constant negative curvature or pseudospherical surfaces, but was also interested in the material construction of these surfaces. From the correspondence with Jules Hoüel, we learn that he began to develop this interest starting in 1869:

[18] C. Segre to F. Klein, Torino 11 May 1887, in Luciano and Roero 2012 [24], p. 146.

[19] In Severi's *Vorlesungen über algebraische Geometrie*, Leipzig: Teubner, 1921 at p. 215 a model of a particular Riemann's surface is presented.

"*In this period I have had a whimsical idea, which I shall tell you about... I wanted to try to construct materially the pseudospherical surface, on which the theorems of non-Euclidean geometry are fulfilled... If you consider the surface lying between two meridians, close enough together to allow it to be replaced, over a certain length, by a plane, you can, with little bits of paper cut into appropriate shapes reproduce the curved trapeziums whose true surface can be compounded*".[20]

With the aid of a pupil, Beltrami built several cardboard models, one of which is still preserved at the Institute of Mathematics of the University of Pavia. This is the copy [21] which he sent as a gift to his friend Luigi Cremona (1830-1903) on 25 April 1869 with a cover letter where he mentioned, amongst other things, a possible industrial-type production:

"*I wouldn't want the model to remain at the Institute just to become food for the mice ... I would rather that it remain with you, in case one of your well-intentioned students with more patience than mine wished to try a more successful construction than the rough one I made. I also have a few other ideas for a more perfect execution to be made with other means and very different materials: but for this it is necessary to consult someone who is expert in industrial manipulations*"[22].

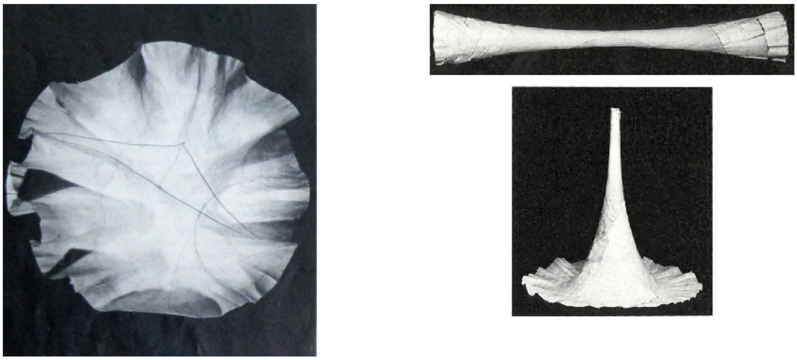


Fig. 1 a, *left*) Beltrami's model; b, *top right*) the model folded to represent the hyperbolic type of pseudospherical surface; c, *bottom right*) the model folded to represent the parabolic type.

This cardboard model (Fig. 1), as Beltrami explains to Cremona, can be folded to represent the hyperbolic type of pseudospherical surface, or the

[20] See the letter from Beltrami to Hoüel, Bologna 13 March 1869, in Boi, Giacardi, Tazzioli 1998 [11], p. 80.

[21] This model is described by Beltrami in the letter to Hoüel, Bologna 22 April 1869, in Boi, Giacardi, Tazzioli 1998 [11], p. 91.

[22] See the letter from Beltrami to Cremona, Bologna 25 April 1869, in Boi, Giacardi, Tazzioli 1998 [11], pp. 201-203.

parabolic type or simply the pseudosphere, but it cannot be folded to represent the elliptical type of pseudosphere without making a cut. For Beltrami it was not only a pleasurable pastime, but a tool for experimentally verifying the results obtained with theory and for visualising properties, as well as an artefact for discover new properties.

For example, one new result that he obtained handling his cardboard model is the following (Fig. 2):

"*Draw a straight line AB and at each of its points M draw the straight line MT which marks the direction of the parallel to AE, following Lobachevsky, with respect to the distance AM. The envelope of these straight lines is the meridian of the pseudospherical surface. The distance MN to the point of contact is constant*"[23].

Beltrami published this result later in 1872.[24]

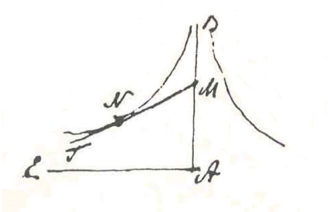


Fig. 2

Another important result intuited by Beltrami through manipulating his cardboard model was the following that he communicated to Hoüel:

"*You speak of empirical propositions that can be found by this means [the model], and you are perfectly correct, in fact here we have surfaces for which we possess no general equations. Here then is a proposition that I have begun to intuit: A pseudospherical surface can always be folded such that any whatsoever of its geodesic lines becomes a straight line. I give this to you only as an approximate result that is produced when, holding firm with both hands two points of the flexible surface, it is stretched as far as possible without tearing it. This result was even more striking to me because I supposed the opposite (I cannot say now on the basis of what arguments*"[25].

The last evidence of Beltrami's interest in the material construction of pseudospherical surfaces is the 1872 article *Sulla superficie di rotazione che serve di tipo alle superficie pseudosferiche*, aimed, as he said himself, "to prepare the geometrical elements–possibly easy and exact–of a material construction of the surface itself"[26]. After 1872 Beltrami abandoned this type of research

[23]Letter from Beltrami to Hoüel, Bologna 13 March 1869, in Boi, Giacardi, Tazzioli 1998 [11], p. 82.

[24]E. Beltrami 1872, Teorema di geometria pseudosferica, *Giornale di matematiche*, 10, p. 53: *Opere matematiche*, Milano: Hoepli, 1902-20 (4 vols.), II, 392-393.

[25]Letter from Beltrami to Hoüel, Bologna 25 March 1869, in Boi, Giacardi, Tazzioli 1998 [11], p. 86.

[26]E. Beltrami 1872a, Sulla superficie di rotazione che serve di tipo alle superficie pseudosferiche, *Giornale di Matematiche* 10, pp. 147-159, cit. at p.147; *Opere matematiche* II, 394-409.

and the correspondence with Hoüel shows this fact clearly, because afterwards no mentions to the construction of models appeared. His interests had progressively shifted toward issues regarding mechanics and mathematical physics. When Beltrami told Cremona about his attempts to build a concrete model of pseudospherical surface, he wrote:

"*The news that you are involved with the material construction of the pseudospherical surface pleases me greatly. These effective constructions are one of my dreams: but I don't know where to begin. I thus wait eagerly for you to communicate your results to me*"[27].

Cremona is the well-known mathematician, who developed the theory of birational transformations, but also the promoter of the return to the use of Euclid's *Elements* as a textbook in classical schools and therefore a supporter of a rigorous teaching of geometry. Notwithstanding, rigor was not the only aspect that characterized his vision of teaching: he also took in account *dynamic* aspects (based on the idea of transformation), creative aspects (which referred to the faculties of imagination, and the aesthetic sense), to which the historical aspect must be added[28]. Despite this, as far as we know, it does not seem that Cremona built geometric models, but he was certainly interested in them. In addition to the letter to Beltrami this curiosity is attested by some letters to Thomas Hirst [29], where he inquired him about a model that Sylvester intended to build, or on models built by Julius Plücker, or told about his visit in Munich during which Klein and Brill showed him the collection of geometric models. In the letters exchanged with Plücker Cremona asked him if there were photographs of the models built by Epkens for the Exhibition of Paris in 1867 [30]. It is quite probable that the models and their use were also the topic of correspondence with Klein: in 1869 in a letter to Cremona, Klein underlined the importance of models:

"*To me [this] section seems to be interesting, even apart from the theory of complexes, in so far as here a diverse family of surfaces is discussed in such a manner that the various shapes which arise are made evident. It has always seemed to me–and in this sense I understand Plücker's method when he had*

[27] Letter by Cremona to Beltrami, Milano, 25 March del [1869]. The letter can be found on the website www.luigi-cremona.it. Beltrami kept a dense correspondence with Cremona: 1036 letters from 1864 to 1900.

[28] An example of this point of view is the article L. Cremona 1860, "Considerazioni di storia della geometria in occasione di un libro di geometria elementare pubblicato a Firenze", *Il Politecnico*, 9, pp. 286-323 (*Opere matematiche* I, 176-207). See also Brigaglia 2006 [17].

[29] See the letters from Cremona to Hirst: Bologna 18 January 1965, St. John Wood 13 February 1865, Bologna 24 March 1865, Milan 3 March 1868, 5 Oct 1876, in Nurzia 1999, pp. 72, 75, 77, 137, 175.

[30] See the letters between Plücker and Cremona: Plücker to Cremona, Bonn 31 March 1867, Bonn 7 August 1867, Cremona to Plücker, Milan 5 April 1867, in Millán Gasca 1992 [29], pp. 154-155, and Menghini 1994 [27], p. 74.

models constructed of the surfaces he dealt with there–that for geometrical problems not only are the theorems important that express relations between the objects to be treated but also important is the direct visualisation of these objects"[31].

Furthermore, among the papers left by Cremona's daughter Itala Cozzolino to the Istituto Mazziniano in Genoa, there are beautiful cardboard models that Brill sent as gifts to Cremona, as it appears from the original envelope. They came from the first collection sold by Brill (Fig. 3)[32].

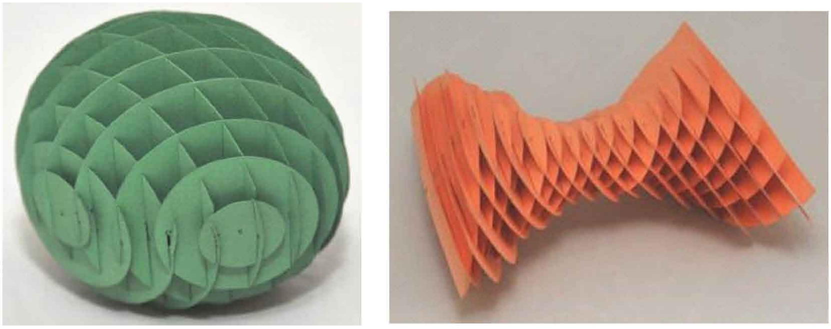


Fig. 3 Cardboard models of an ellipsoid and of a hyperboloid that belonged to Cremona

3 Models in the teaching of geometry at secondary schools at the end of the nineteenth century

Actually, in Italy the use of models had spread in mid-nineteenth century in the secondary schools and in the training of primary school teachers, in connection with the pedagogical movement promoted in Torino by the educators Ferrante Aporti, the founder of the first kindergartens (*asili aportiani*), Vincenzo Troya and Antonio Rayneri[33].To go beyond the catechistic and repetitive methods in use up until that time, they maintained the importance of adopting, especially in primary teaching, the Socratic and intuitive methods, and the usefulness of basing teaching on children's experience and the manipulation of concrete objects[34]. In his book for primary teacher training, *Lezioni di nomenclatura*

[31] Letter from Klein to Cremona, Göttingen, 6 July 1869, in Menghini 1994 [27], p. 55. Other letters from ad to Klein are kept in the Istituto Mazziniano in Genoa and can be found in: http://www.luigi-cremona.it/, but they do not contain anything concerning the geometric models. I am grateful to Simonetta di Sieno for having provided me with these letters.

[32] See Rowe 2013 [36].

[33] On the historical context see Pizzarelli 2013 [34].

[34] See C. Sideri 1999, *Ferrante Aporti, Sacerdote, italiano, educatore*. Milano: Franco Angeli, pp. 69-107.

geometrica (Torino: Paravia, 1851, II ed. 1952) Rayneri explicitly discusses the use of geometric models (Fig. 4), stating:

"*It is to be desired that all students can observe them at their leisure, collocating them in diverse positions and comparing their various parts to each other and this is impossible if the objects are not submitted to a direct observation*". (p. XXXVI, II ed.)

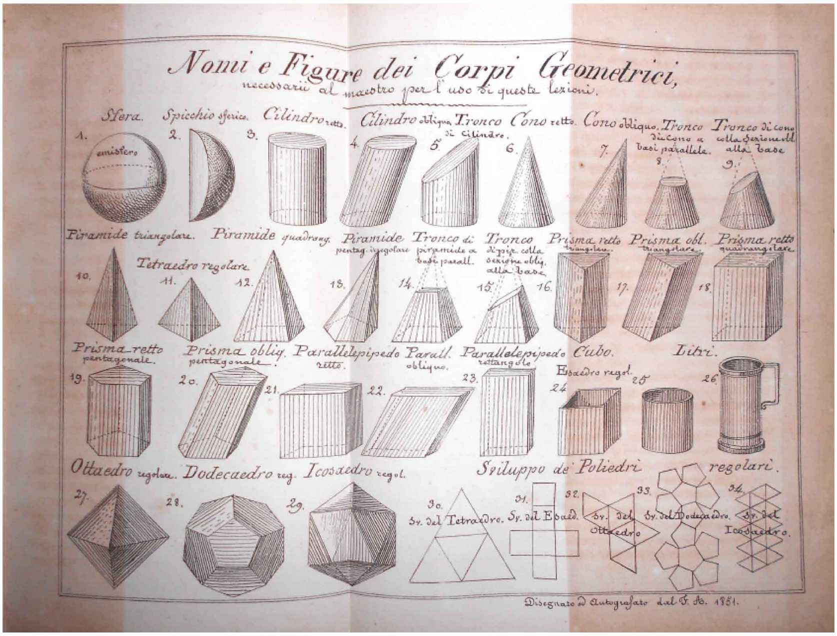


Fig. 4 Plate from Antonio Rayneri, *Lezioni di nomenclatura geometrica* (1851)

Under Rayneri's supervision, Giuseppe Crotti, professor of geometry and applied mechanics at the night school of the Workers Society in Torino, assembled two collections of geometric solids, one with 27 exemplars and the other of 35, sold in three sets (large, medium and small) by Paravia, a printer and bookseller in Torino, which at the time had begun to market educational aids of various kinds for schools[35].

In the years that followed other publishing houses also began to publish catalogues of school materials, including collections of geometric models for classroom use. One such publisher was Giacomo Agnelli of Milan, whose catalogue for 1890-1891 presented a collection–available in six sets, including

[35] See *Giornale della Società d'Istruzione e di Educazione*, IV, 1852, pp. 109-110; see also the catolog by Paravia *Elenco di libri ed oggetti per le scuole normali-magistrali, elementari, tecniche, ginnasiali e liceali*, Torino, Milano: Paravia e Comp., 1862, p. 5.

one of *stragrandissimi* (gigantic sizes)–of 24 demountable geometrical solids (Fig. 5)[36]. This free catalogue, of which 25,000 copies were printed, was sent to all schools.

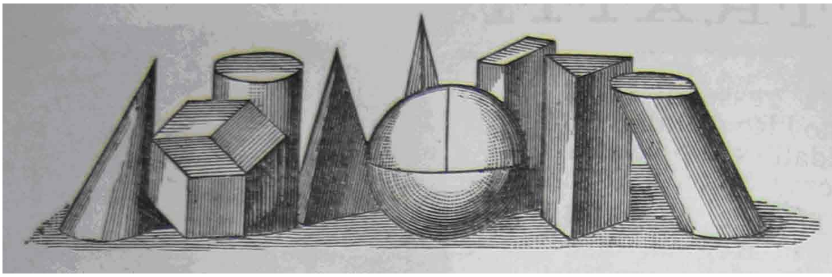


Fig. 5 The collection of demountable solids advertised in the Agnelli catalogue of 1890-1891

Fig. 6 The collection of crystallographic models conserved today at the Istituto Tecnico Cavour in Vercelli

Also particularly famous was the collection of crystallographical models conceived by Quintino Sella (1827-1884), one of the founders of mathematical crystallography, when he was teaching geometry applied to the arts in the 1850s at the Istituto Tecnico in Turin and later used in his classes in crystallography at the Scuola di Applicazione degli Ingegneri (the engineering school that later became the Politecnico di Torino)[37]. The models were built of wood by Giovanni Blotto under Sella's direction in two series of 200 models each, with one series coloured and one not.[38] Some of these crystallographic models are today conserved at the Istituto Tecnico Cavour in Vercelli (Fig. 6).

The use of models for the teaching of geometry was prescribed by the school legislation: for example Article 152 of the 1853 Regulations for Schools for Teachers in primary and special schools of the Kingdom of Sardinia specified that such schools were to have geometric solids as part of their

[36] *Libri ed articoli scolastici approvati per le Scuole del Regno*, Milano: Ditta Giacomo Agnelli, 1890-91, p. 84.

[37] See Q. Sella, *Lezioni elementari di Cristallografia dettate alla Scuola d'Applicazione degli ingegneri di Torino nel 1861-62*, Torino: Briola 1867 (lithograph).

[38] See footnote 48. See also G. Blotto, *Catalogo dei modelli in legno di meccanica, costruzioni e cristallografia*, Torino: R. Scuola d'Applicazione per gli Ingegneri, 1869, pp. 22-30.

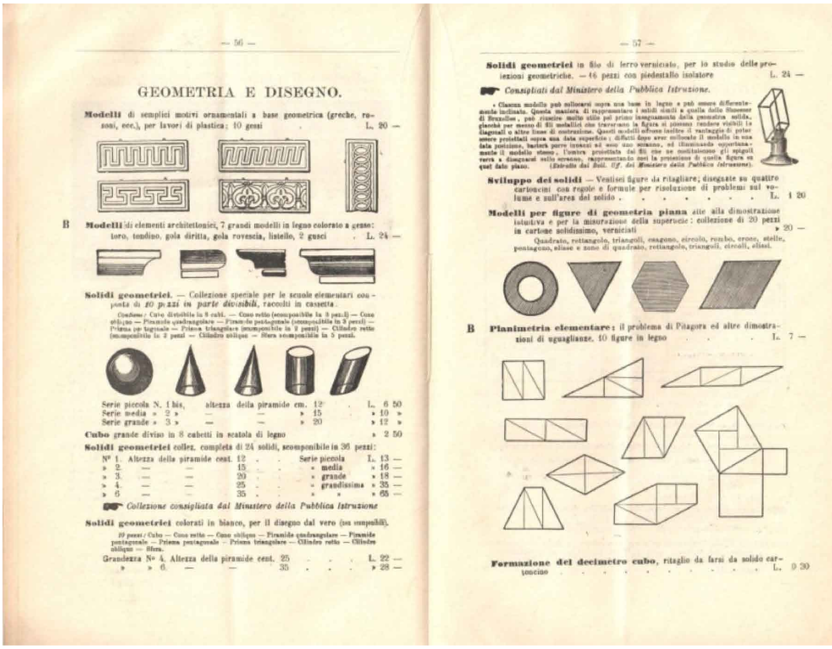


Fig. 7 Catalogue of school material for pre-schools and elementary schools (Paravia, 1913-1914, p. 57).

equipment[39], and Art. 55 of the 1860 Regulations of the Scuola di Applicazione degli Ingegneri in Turin appear to indicate that the "model-making workshop" of this School was given permission to build models for teaching for secondary schools in Piedmont[40]. In the Regulations (1867, 1892, 1895)[41] for *Scuole normali* (schools for elementary teachers training) of the Kingdom of Italy the use of models was recommended: in particular in the 1892 and 1895 Regulations, teachers were invited to "have the students construct the figures with cardboard, wire, etc., in order to better *derive a model from the drawings* done on the blackboard"[42]. Mathematics programs of 1881, 1885 and 1890 for technical schools also invited teachers to use "large scale" models of geometric solids for teaching geometry[43]. In the Paravia catalogues of school materials there

[39] *Raccolta degli atti del governo di Sua Maestà il Re di Sardegna*, 1853, N. 1599, p. 1134.

[40] *Raccolta degli atti del governo di Sua Maestà il Re di Sardegna*, 1860, N. 4338, p. 1913, 1916. See also L. Sassi 1996, Rapporti istituzionali e legami culturali fra le scuole politecniche superiori e gli istituti tecnici e professionali secondari nel Piemonte post-unitario, *Le culture della tecnica*, 1, pp. 89-105, at p. 105.

[41] *Raccolta ufficiale delle leggi e decreti del Regno d'Italia*, 1867, vol. VII p. 256; 1892, vol. IV p. 3622; 1895, vol. IV p. 4245.

[42] *Raccolta ufficiale delle leggi e decreti del Regno d'Italia*, 1895, vol. IV p. 4280.

[43] *Raccolta ufficiale delle leggi e decreti del Regno d'Italia*, 1881, 1885 and 1890, in *Documenti*

appeared models of geometric solids first in wood or wire, and then also in cardboard[44], and models of solids for teaching descriptive geometry appeared in collections that were increasingly varied and more beautiful up until the 1960s, often accompanied by the line "Collection recommended by the minister for public instruction"; for example, the catalogue of 1913-1914, in the section for "geometrical solids in painted wire, for the study of geometrical projections" carried an excerpt from the official bulletin of the Ministry for Education which underlined the benefits of the collection (Fig. 7):

"*The models further offer the advantage that they can be projected onto a surface: in fact, after the model has been placed in a certain position, it is sufficient to place a screen in front of it, and by properly illuminating the model itself, the shadow projected by the wires that make up its edges, will be drawn on the screeen, thus representing the projection of that figure on that given plane*"[45].

Fig. 8 Cover of the Vallardi catalogue of 1822-1823.

In the course of the twentieth century, other publishing houses in addition to Paravia also began to issue catalogues of school materials and distribute them by the thousands to Italian schools. One example was Antonio Vallardi of Milan (Fig. 8), which in the 1960s also added to its catalogue models of geometric solids in plastic[46]. Another example is Arnoldo Mondadori of Verona, which began to produce this kind of catalogue in 1927-1928 and also

per la storia dell'insegnamento della matematicain Italia (ed. by L. Giacardi and R. Scoth): $http://www.mathesistorino.it/?page\_id = 564$.

[44]See, for example, *Catalogo del materiale scolastico e dei sussidi didattici per le scuole elementari*, Torino, Milano, Padova, Firenze, Napoli, Roma, Catania, Palermo: Paravia, 1937-1938, pp. 62-66.

[45]See *Catalogo del materiale scolastico per gli asili infantili e per le scuole elementari,* Torino, Roma, Milano, Firenze, Napoli, Palermo: Paravia, 1913-1914, p. 57.

[46]See *Scuole elementari. Materiale didattico*, 8, Milano: Vallardi, 1961.

set up showrooms for this sort of materials in its offices in Verona, Rome, Milan and Turin[47].

It is thus natural that the pedagogical congresses and the various national exhibitions often featured displays of models for use in secondary schools, as part of the section for education. For example, there were sections dedicated to education in the National Exhibition of 1858 in Turin, where two collections (of 200 pieces each) of Sella's wooden crystallographical models were displayed[48], in the Pedagogical Congress held in Turin in 1869[49], in the Italian Industrial Exhibition of 1884 in Turin[50] and in the National Exhibition of 1891 in Palermo, where solids made by the students of the Industrial School "Alessandro Volta" of Naples were exhibited[51].

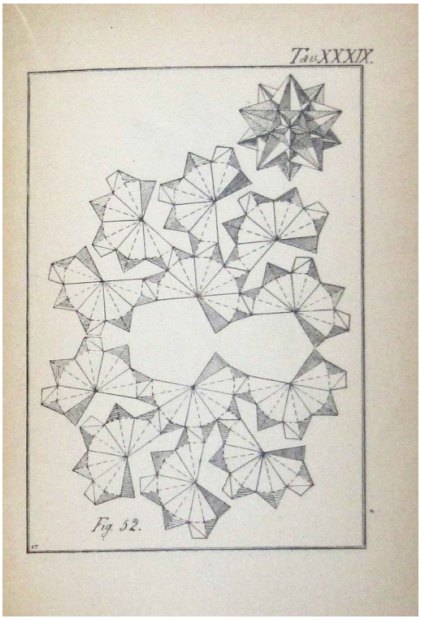


Fig. 9 Plate XXXIX from Alfonso Rivelli, *Stereometria applicata allo sviluppo dei solidi ed alla loro costruzione in carta* (1897).

It is worth mentioning that in 1897 the publisher Hoepli in Milan printed the manual by Alfonso Rivelli *Stereometria applicata allo sviluppo dei solidi ed alla loro costruzione in carta* addressed to secondary schools, which contains the explanation of how to build the various fundamental basic solids, but also regular star polyhedra–some of which were quite complicated–and the sphere.

The book is enriched by various exercises which invite the student to apply what he has learned to the construction of new solids.

[47] See *Catalogo materiali didattici*, Verona: A. Mondadori, 1930, p. 9. Various of these catalogues are available in the Museo della Scuola e del Libro per l'Infanzia (MUSLI) in Torino, and the Museum has begun survey of didactic materials used in primary schools in Turin starting in the mid-1800s.

[48] See *Relazione dei giurati e giudizio della R. Camera di Agricoltura e Commercio sulla Esposizione Nazionale di prodotti delle industrie, seguita nel 1858 in Torino*, Torino: Stamperia dell'Unione Tipografico-Editrice, 1860, p. 44.

[49] See R. *Museo industriale italiano. Illustrazioni delle collezioni.* Didattica, Torino, Napoli: dall'Unione Tipografico-Editrice, 1869.

[50] La mostra didattica, Torino. *L'Esposizione italiana* 1884, N. 30 Torino, Milano: Roux e Favale.

[51] A. Rivelli, *Stereometria applicata allo sviluppo dei solidi ed alla loro costruzione in carta*, Milan: Hoepli 1897, p. 10.

The representation of geometric models and their nets also appeared in several geometry textbooks for the lower level of secondary schools. For example, Giuseppe Veronese in his book *Nozioni di geometria intuitiva* (Verona: Drucker 1908) shows how to build the regular polyedra beginning with their nets cut out of cardboard (pp. 94-98) because in this way, constructing and then handling models, the student is not obliged to follow passively the reasoning of the teacher, but can play an active part in the process of learning (p. VI). In the textbook by Angelo Pensa, *Elementi di geometria ad uso delle scuole secondarie inferiori* (Torino: Petrini, 1912), solids are represented with their nets, but the treatment is static and does not make any mention of the actual construction of models.

4 Corrado Segre and the use of models at the University of Turin

As already mentioned, the most important Italian universities bought collections of models from abroad. In Turin the first acquisitions date to 1880-1881, thanks to Enrico D'Ovidio (1843-1933), professor of higher geometry, rector of the University, director of the Teacher Training School and from 1883 director of the Mathematics Library[52].The first to be purchased were surface models, in cardboard, plaster, wire, to be used for educational purposes. Documents from the archives of the "Giuseppe Peano" Library of Mathematics, tell us that, in January 1882, 47 plaster models by Brill of the first series I-VI, 7 cardboard models of quadric surfaces, 8 wireframe models by Björling at a total cost of 1,265.60 Italian lire, were purchased[53]. In the years that followed, the models acquired by the Mathematics Library in Turin came mainly from the catalogues of Brill and of Schilling. Purchases are documented until 1919[54]. Among these also figure the five regular solids produced by Paravia. The documentation regarding the periodical restorations (by Fratelli Pallardi, Turin) of the models testify to the customary use of these in classes of advanced geometry [55] and in the mathematics lectures for the Teacher Training School. In November 1907 Corrado Segre (1863-1924) took over the direction of the Library from D'Ovidio and held that position until his death. Segre–the founder

[52]E. D'Ovidio, Relazione delle cose più notevoli accadute durante l'anno scolastico 1880-81 nella R. Università di Torino, *Annuario, Università di Torino*, 1881-1882, pp. 3-7, cit. p. 7.

[53]See *Inventario... dal 1 aprile 1881 al 31 marzo 1883 in Dossier Inventari Consorzio*, Library of mathematics "Giuseppe Peano", Turin.

[54]See Giacardi 2003 [20] and Ferrarese 2004 [8].

[55]See, for example Segre's notebooks *Applicazione degli integrali Abeliani alla Geometria* (1903-04), p. 26 and *Superficie del* 3^o *ordine e curve piane del* 4^o *ordine* (1909-10), p. 176, Fondo Segre, Library of Mathematics "Giuseppe Peano", Turin, Quaderni 17 and 23, now in http://www.corradosegre.unito.it/I11_20.php.

of the Italian School of Algebraic geometry that numbers among its members outstanding mathematicians such as Guido Castelnuovo, Francesco Severi, Federigo Enriques, Gino Fano, Beppo Levi, Alessandro Terracini and Eugenio Togliatti–increased the collection of models, which he used in both his courses in higher geometry and in his lectures for the Teacher Training School:

"*Corrado Segre gave classes on Tuesday, Thursday and Saturday mornings from 10 to 11, originally on the first floor in the lecture hall that occupied the place then used as the foyer of the Aula Magna, and later, I believe, in that lecture hall XVII of the second floor of the University building in Via Po, whose walls were lined with the glass cases with Brill's geometric models*"[56].

In fact, like Beltrami, Segre believed that the models can sometimes smooth the path to discovery, making it possible to "see certain properties that with deductive reasoning alone cannot be obtained".[57] This conviction derived

Fig. 10 Some of the models from the "Giuseppe Peano" Mathematics Libray of the University of Turin (Kummer surface, pseudosphere, Cayley surface, helicoid).

directly from his way of conceiving scientific research, in which geometric intuition played a significant role, as he wrote to Klein:

"*... the method that I like most, by my scientific inclinations, is the one mainly due to you: the geometric, or rather, synthetic, because it makes use of ingenious reasoning rather than lengthy calculations*"[58] .

This view also influenced his way to conceive the teaching of mathematics in secondary schools, as it clearly stands out from the manuscript notebook[59] of the lessons that for many years (from 1887/88 to 1890/91 and then again from 1907/08 to 1919/20) he taught at the Teacher Training School of the University of Turin. During his classes, in addition to dealing with elementary mathematics from an advanced standpoint, he also addressed didactic and

[56] Terracini 1968 [39], p. 10.

[57] Segre, 1891 [37], p. 54.

[58] Letter from Segre to Klein, Torino 7 October 1884, in Luciano, Roero 2012 [24], p. 134.

[59] See C. Segre, [Appunti relativi alle lezioni tenute per la Scuola di Magistero], Fondo Segre, Library of mathematics "Giuseppe Peano", Turin, Quaderni 40 now in http://www.corradosegre.unito.it/I31_40.php. On this subject see Giacardi 2003a [20].

methodological questions and, in particular, he believed that the first approach to mathematics should be experimental and intuitive, so that the student learns "not only to demonstrate truths already known, but to make discoveries as well, to solve the problems on his own"[60]. As far as geometry is concerned, Segre agreed with Giovanni Vailati's point of view, according to which the teaching of this discipline should be experimental and operative and benefit from teaching aids such as squared paper, drawing, and geometric models. Segre's notebook includes an annotated bibliography, where he lists not only papers by Vailati and Marcolongo, but also the little book by Rivelli mentioned above and that by Karl Giebel, *Anfertigung mathematische Modelle für Schüler mittlerer Klassen* (Leipzig: Teubner 1915) on the construction of models for secondary schools teaching. With the aim of offering teachers a book inspired by this method, Segre invited one of his students, Luisa Virgilio, to translate Grace and William Young's *A First Book of Geometry* (1905), in which the discovery of geometrical properties and theorems arises from the construction and manipulation of cardboard models. By folding paper, the authors guide the student towards simple "proofs" of some fundamental propositions. Then, they explain in detail how to build the regular polyhedra–or solids obtained from these by truncating them or placing two of them together –with appropriate cardboard "plane models" (with auxiliary faces), so that the student

"*acquires not only manual dexterity, but complete familiarity with the truths which each model is meant to represent. Just because he can do this by himself, he is not taught, but learns, and he develops what may be called his geometrical sense*"[61]

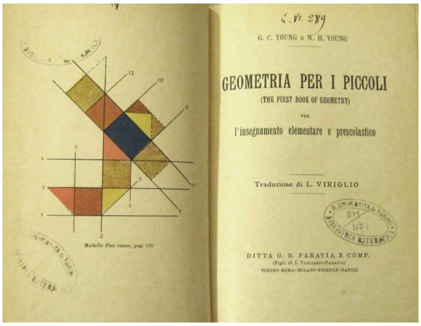


Fig. 11 Grace and William Young, *Geometria per i piccoli* (translation by L. Viriglio), Torino: Paravia 1911.

The use of folding or cutting paper as part of teaching geometry was recommended by educators such as Pietro Pasquali (1847-1921), who believed in the educative value of manual work and maintained that the kind of teaching addressed to children should be fun and spontaneous. His publications included *Geometria intuitiva, ad uso delle scuole elementari superiori, tecniche, normali e industriali. Lezioni di ritaglio geometrico* (Parte I, Parma: Luigi Battei, 1901; Parte III, Milano: A. Vallardi, 1906)[62].

[60] Ibid., pp. 15, 16, 42.
[61] G.C. Young, W.H. Young, *A First Book of Geometry*, London: J. Dent, 1905, p. Viii.
[62] See for example G. Vacca 1930, Della piegatura della carta applicata alla geometria, *Periodico*

5 The role of models in the "laboratory of mathematics"

The book by Grace and William Young was inspired by a laboratory vision of mathematics teaching. The idea of a laboratory of mathematics[63] was introduced in the late nineteenth century by John Perry (1850-1920), a professor of mechanics and mathematics at the Royal College of Science in London, who proposed a new teaching method that he called 'Practical Mathematics', with emphasis on experiments, measurements, data gathering, drawing, graphic methods, and on the relationships between mathematics with physics and other sciences. With regard to geometry, Perry criticised the Euclidean method and suggested a teaching that was practical and experimental, heavily based on drawing, measuring and the use of squared paper. However, in his most famous book *Elementary Practical Mathematics* (London: Macmillan, 1913) no mention is made of geometric models.

Instead, in France, after the secondary school reform of 1902, Emile Borel (1871-1956) together with Jules Tannery (1848-1910) created the Laboratoire d'enseignement mathématique at the Ecole Normale Supérieure. This laboratory was aimed at training future teachers: here models in either wood or cardboard, wire and cork were conceived and built for teaching geometry and mechanics. The didactic uses of other instruments such as mechanical linkages, pantographs, inversors, calculating machines, and instruments for geodesy and land surveying were also taught[64]. The stage of conceiving and then constructing models was a significant one, as Borel himself affirmed[65].

Mentions of the use of models in secondary teaching can be found in the Meraner Lehrplan (1905)[66], prepared by the Unterrichtskommission der Gesellschaft deutscher Naturforscher und Ärzte (the Teaching Committee of the German Society of Natural Scientists and Physicians), adopting some of the basic points of Klein's reform movement. With regard to the teaching of geometry, the following aspects were emphasised: the strengthening of spatial intuition (p. 543); the use of straightedge and compass, drawing, measuring (p. 547); the consideration of geometrical configurations as dynamic objects (p. 548); making room for applications (p. 549); making use (*Benuzung*) of

di Matematiche, s. IV, X, pp. 43-50.

[63] For more on this subject see Giacardi 2011 [22].

[64] See A. Châtelet 1909, Le laboratoire d'enseignement mathématique de l'Ecole Normale Supérieure de Paris, *L'Enseignement mathématique*, 11, pp. 206-210.

[65] É. Borel 1904, Les exercices pratiques de mathématiques dans l'enseignement secondaire. *Revue générale des sciences pures et appliquées*, 15, 431-440, in Gispert 2002 [23].

[66] See Bericht betreffend den Unterricht in der Mathematik an den neunklassigen höheren Lehranstalten, *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht*, 36, 1905, pp. 543-553. Also in F. Klein 1907, *Vorträge über den mathematischen Unterricht*, Teil 1, Leipzig: Teubner, pp. 208-219. See the English translation of the curriculum of Gymnasia in *The Mathematical Gazette*, 6. 95, 1911, pp. 179-181.

models; the coordination of planimetry and stereometry (p. 550). The importance given by Felix Klein (1849-1925) to geometric models as an *Anschauungsmittel* (visual aid) in research and teaching of mathematics is well known and studied,[67] thus I only wish to point out that he frequently underlined the didactic function of models. For example in a lecture given in 1893 in Chicago he affirmed:

"*The principal effort has been to reduce the difficulty of mathematical study by improving the seminary arrangements and equipments (sic)... Collections of mathematical models and courses in drawing are calculated to disarm, in part at least, the hostility directed against the excessive abstractness of the university instruction*"[68].

In his book *Anwendung der Differential-und Integralrechnung auf Geometrie* (Leipzig: Teubner 1907) Klein devoted a chapter, to the *Versinnlichung idealer Gebilde durch Zeichnungen und Modelle*, that is, the *Concretization of ideal objects through drawings and models* (p. 424). Moreover, in his work on elementary mathematics from an advanced standpoint, in the volume on geometry, Klein presented various instruments, recommending that "the actual practical demonstration" not be neglected, when such instruments are considered in illustration of a theory[69].

Klein was also involved in the reorganisation and modernisation of the *Modellkammer* in Göttingen for educational purposes, in particular to foster the *Raumanschauung* (spatial intuition)[70], and at the meeting of the International Commission on the Teaching of Mathematics (later ICMI) held in Brussels in 1910 he presented the geometric models from the collections of Brill and Schilling and illustrated their use in university teaching.[71] During that same meeting Peter Treutlein (1845-1912) presented a new series of models that were about to be released by the Teubner publishing house, and showed their use in secondary school[72]. In the catalogue–published two years later under the title *Verzeichnis mathematischer Modelle Sammlungen H. Wiener und P. Treutlein* (Leipzig: Teubner, 1912)–Treutlein devoted a section to models for teaching plane and solid geometry in secondary schools.

In Italy it was Giovanni Vailati (1863-1909), a mathematician, a philosopher and a member of the Peano School, who proposed an active approach to the teaching of mathematics, in the context of the work performed for the Royal Commission for the reform of secondary schools (1905-1909). He named this

[67] See Rowe 2013 [36].
[68] F. Klein 1894, *The Evanston Colloquium Lectures on Mathematics*, New York: Macmillan and Co, pp. 108-109.
[69] F. Klein 1925, *Elementarmathematik vom höheren Standpunkte aus, II Geometrie*, Berlin: Springer, p. 16.
[70] For more on this, see Bartolini Bussi et al. 2010 [15] and Schubring, 2010 [38].
[71] See *L'Enseignement mathématique,* 12, 1910, pp. 391-392.
[72] Ibid., p. 388.

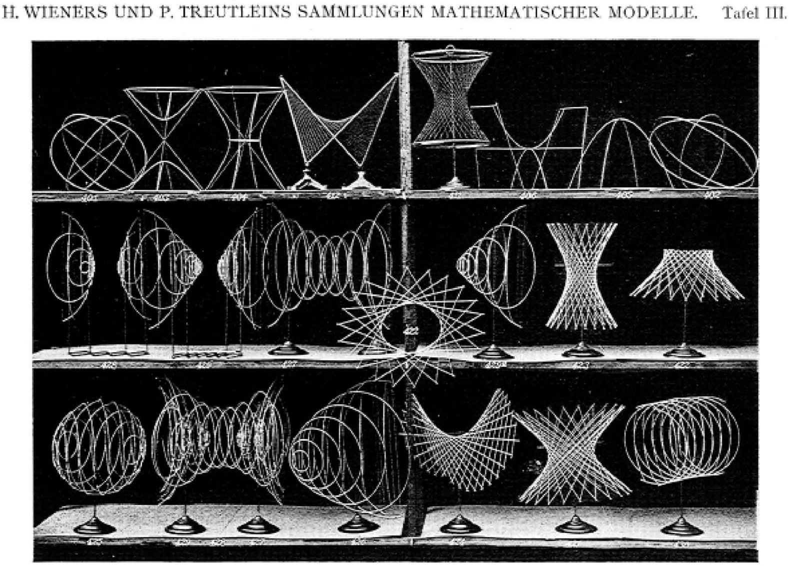


Fig. 12

approach 'school as laboratory', not in the reductive sense of a laboratory for scientific experiments, but "as a place where the student is given the means to train himself, under the guidance and advice of the teacher, to experiment and resolve questions, to ... test himself in the face of obstacles and difficulties aimed at provoking his perspicacity and cultivating his initiative"[73]. In particular, according to Vailati, in the teaching of geometry the teacher should adopt an approach that is *sperimentale–operativo* (experimental and active), using squared paper, drawing and geometric models, that is an approach directed "at developing not only the students' skills of observation but also those that come into play in the construction of the figures and in comparisons between them and their various parts, by means of measures, decompositions, movements, in short, by means of all those procedures that can lead to affirming and verifying their properties, which will later form the object of analysis and proof[74].

[73]G. Vailati 1906, Idee pedagogiche di H. G. Wells, in M. Quaranta (ed), *Giovanni Vailati, Scritti*, 3 vols., Bologna: Forni, 1987, III, pp. 291-295, at p. 292.

[74]G. Vailati 1909, Sugli attuali programmi per l'insegnamento della matematica nelle scuole secondarie italiane. In *Atti del IV Congresso Internazionale dei Matematici*, 6-11 aprile 1908, Roma: Tip. Accademia dei Lincei, 482-487, pp. 484-485. It is interesting to note that in the questionnaire that the Royal Commission sent to all schools before formulating its project for the reform of secondary schools, there is a specific question about whether or not the school owned a collection of models of geometric solids (see *Risposte al questionario diffuso con circolare 27 marzo 1906*, II vol. of *Commissione Reale per l'ordinamento degli studi secondari in Italia*, 2 voll., Roma: Tip. Cecchini, 1909, p. 48).

So Vailati, like the Youngs, conceived a dynamic use of models in the teaching of geometry. Various factors which cannot be further analysed here prevented the mathematics laboratory proposed by Vailati from becoming widespread in practice. First of all, the reform set forth by the Royal Commission was never carried through, second, Vailati never wrote a systematic exposition of his ideas and his premature death prevented any further developments. Third, laboratory method-inspired textbooks were never published in Italy, even though some authors paid attention to the geometrical constructions and to the experiments with folded or cut paper, sand, or small models in geometry. Fourth not all mathematicians in Italy shared Vailati's approach to teaching of mathematics and finally, his efforts would have been in any case nullified by the Gentile Reform of 1923, which made the humanities the cultural axis of national life in Italy, and especially of education (Giacardi 2011).

The only one who took up the idea of an effective laboratory-type teaching of mathematics at secondary schools was Marcolongo who, as seen above, had contributed to enrich the collection of models and mathematical instruments of the Rational mechanics Institute of the University of Naples. During the National Congress of the Mathesis (national association of mathematics teachers), held in Naples in October 1921, Marcolongo set up an exhibition of models and instruments, some of which were built by teachers or by students of secondary schools. Among others, wood or cardboard models of the principal elementary solids, the regular polyhedra, Dupin's decomposable solids, Brill's cardboard models of surfaces, Vuibert's anaglyphs, geometric figures for the stereoscope, and more were displayed[75]. Marcolongo also gave a lecture (Marcolongo 1922) on the use of educational and experimental materials in teaching practice, in which he illustrated his "ideal laboratory of mathematics". As he stated himself, he always had a passion for models and believed that the geometric experiments made by students through drawing and use of models "can not only promote geometric invention, and the discussion of problems, but also give a sense of confidence and mastery of the subject that is difficult to acquire by other means" (p. 7). In a secondary school laboratory of mathematics, according to Marcolongo, there should be first of all, the models of solids of the elementary geometry and there should also be stereoscopes and stereoscopic figures useful to simulate three-dimensional vision and geometric anaglyphs by Henri Vuibert, that substantially create virtual geometric models. Moreover he was convinced that models had to be built by the students themselves in order to be really useful:

"*Paravia sells, for a modest price, a good collection of solids in wood, widely available but not as equally widely used, in all Italian schools. I have*

[75]The list of models and instruments can be found in Cardone 1996 [13], pp. 148-149.

always had little sympathy with these collections; ...they allow little or nothing of the inside to be seen, ... Worse still, they are not built by students; they represent an experiment already made; instead the experiment must be made by the student, naturally under the guidance of the professor. How much to be preferred are the models in cardboard, those made with thin wooden sticks connected to each other with a bit of wax and fine silk threads, built by the students!" (p. 8)*... in knowing hands, the small model can be an inexhaustible source of simple experiments, easily verified, a starting point for observation and for the experimental discovery of new properties that the student will then attempt to prove by means that are logically rigorous... Make it so the professor comes to his classes armed with models, sheets of paper (even better if colored); that he has [the students] first observe, experiment, and then deduce and ... everyone will see ... a change of scene; the student will grow animated, will understand and (permit me the phrase) digest immediately*" (p. 9).

Marcolongo ended his lecture criticizing Italian mathematicians who too often disdained practical and experimental aspects of their discipline:

"*Our nation has had and still boasts of men of the highest merit both in the field of pure scientific speculation and in the field of pure experimentation; it has many, many fewer examples than other nations of that happy marriage of high scientific speculation with experimental ability that above all England and Germany can boast of*" (pp. 13-14).

He also observed that to reverse this trend it was necessary to start from the secondary schools.

6 The reconstruction of the university collections of models

The golden age for the construction of models in Germany came to an end with World War I, and in the 1930s production gradually ceased altogether, not only for reasons of marketing, but also because of the prevailing of a more abstract point of view in mathematical research[76]. In Italy after World War II there was a revival of interest, due to the fact that in many Italian universities the collections had been destroyed during bombardments. In 1951 the Italian Mathematical Union[77], during its Fourth National Congress in Taormina, promoted the reconstruction of the collections of surface models in plaster or metal wire and Luigi Campedelli (1903-1978), mathematician from

[76]The first model constructed after the war was that of the Peano surface, and even though there were new ones in preparation, their production was postponed due to unfavourable market conditions. In 1932 M. Schilling informed the Mathematical Institute in Göttingen that in the last few years no new exemplars had appeared; see Fischer 1986, p. X.

[77]Modelli per gli insegnamenti di Geometria e di Analisi, *Bollettino della Unione Matematica Italiana*, 3, VI, 1951 [31], p. 366.

the University of Florence, was charged of this initiative. The Mathematics Institute of Pavia made available its rich collection of models from Germany and some artisans in Florence looked after their reproduction[78]. Various Italian universities acquired the collections, and these include the University of Turin which also retains the related documentation. Afterwards models appeared in some university publications such as those of Mario Villa[79] or of Campedelli [80] himself, who, among other things, treated the use of geometric models in the teaching at both secondary schools and university, in a large chapter of the collective book *Le matériel pour l'enseignement des mathématiques* (1958)[81]–the second promoted by la Commission Internationale pour l'Etude et l'Amélioration de l'Enseignement des Mathématiques (CIEAEM)–which marked an important point in the history of mathematics education. Later, gradually models came out from the university teaching and turned into beautiful display objects.

7 Models and a new way of teaching intuitive geometry in secondary school

One of the chapters of the book *Le matériel pour l'enseignement des mathématiques* was written by a young teacher Emma Castelnuovo (1913-2014), one of the daughters of Guido Castelnuovo, under the significant title *L'object et l'action dans l'enseignement de la géométrie intuitive* (The object and the action in the teaching of intuitive geometry). In the 1940s she had been able to put into practice the ideas maintained by Vailati and Marcolongo, in fact she had introduced and experimented in his school, the Scuola Media Tasso (a lower secondary school) in Rome, a new way to teach intuitive geometry, a way that she called constructive to distinguish it from the descriptive one, generally in use up to that time. In Emma's approach the teaching material, drawings, models, are not considered as something static, through which the properties set out by the teacher should be verified, but they must be constructed and handled by the learner and thus become discovery tools. She declared:

[78] See Primo elenco di modelli fatti costruire presso l'Università di Firenze a cura del prof. L. Campedelli, *Bollettino della Unione Matematica Italiana*, 3, VII, 1952, pp. 465-467; see also Modelli geometrici a cura del Prof. L. Campedelli, *Bollettino della Unione Matematica Italiana*, 3, VIII, 1953, p. 229, and Giacardi 2003 [20].

[79] See M. Villa, *Lezioni di Geometria*, Padova: CEDAM, vol I 1965, Tables I-XII and vol. II Tables I-VIII with photographic reproductions of geometric models from the collection of the Istituto di Geometria "L. Cremona" of the University of Bologna, accompanied by very detailed captions.

[80] See L. Campedelli, *Esercitazioni complementari di geometria*, Padova, CEDAM, 1955, Tables I-VIII; *Fantasia e logica nella matematica*, Milano: Feltrinelli 1966, pp. 49-50, Tables IV-VII .

[81] L. Campedelli, *I modelli geometrici*, in *Il materiale per l'insegnamento della matematica*, Firenze: La Nuova Italia 1965, pp. 143-172.

"*We want to emphasize that in any case, the material must be moveable: mobility is what in fact attracts the attention of the child, and that leads from concrete to abstract notions; because the subject of his attention is not the material itself but rather the transformation of the material, an operation that, being independent from the material itself, is abstract*"[82].

In these words of Emma it is possible to find an echo of the ideas of her uncle Federigo Enriques, who wrote:

"*... the construction of a geometric figure requires not only the attitude of passively seeing a model ... but also the capacity to shape a possible model, on which are imposed, a priori, certain conditions: and this kind of constructive activity which orders the data of observations and past experience, is not pure imagination ... but rather true logical activity*"[83] .

Emma also makes reference to Maria Montessori (1870-1952), the physician and educator who, at the beginning of the century, had introduced the use of special materials for teaching mathematics in primary schools (rods of different lengths, cubes, prism, etc.), and who in 1934 published the two books *Psico-aritmética* (Barcelona: Araluce, 1934) and *Psico-geométria* (Barcelona: Araluce, 1934), in which these materials were used in such a way as to permit, as Emma commented, not only a "grasp of various mathematical concepts by means of the senses, but also a grasp of operations"[84]. This is therefore an intuitive teaching based on a notion of intuition that is not only the "passive perception of an image or a material, but is also construction"[85].

Emma, however, also observed that "something is missing" for arriving to the intuition proper of a mathematician because with the Montessorian materials the mathematical experience is not practised on phenomena that vary

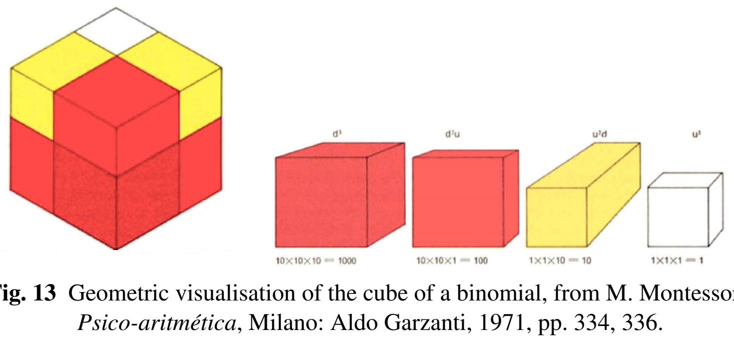


Fig. 13 Geometric visualisation of the cube of a binomial, from M. Montessori, *Psico-aritmética*, Milano: Aldo Garzanti, 1971, pp. 334, 336.

[82] E. Castelnuovo, L'oggetto e l'azione nell'insegnamento della geometria intuitiva, Ibidem, pp. 41-65, at p. 58.

[83] F. Enriques, Insegnamento dinamico, *Periodico di Matematiche* 4, 1, pp. 6-16.

[84] E. Castelnuovo, L'insegnamento della matematica nella scuola preelementare e elementare, *Scuola e Città*, 3, 1957, pp. 93-98, at pp. 94-95.

[85] Ibid., p. 95.

with continuity. A peculiar characteristic of Emma's "intuitive geometry" is in fact the use of moveable models that can show the transformation from one figure to another, and thus stimulate mathematical intuition. Examples that can be cited are the parallelepiped constructed with small jointed rods[86], and the cylinder built of elastic wires that can be transformed into a cone[87]. These are exercises that are possible today through the use of dynamic geometry software. Emma was able to transform his classroom in a laboratory where everybody can perform free individual and collective creative work that trains both the mind and the character, anticipating some of the theoretical elaborations concerning the laboratory of mathematics that have characterized research in education in recent decades[88].

8 A few conclusions

Further investigations into the archives of the Teacher Training Schools and the collections of the oldest technical institutes and classical high schools, a more extensive examination of unpublished scientific correspondence, manuscript lessons, textbooks, treatises, etc., could perhaps provide more accurate evidence of the use of models in mathematics teaching. However, the research carried out up to now concerning the period in consideration, allows us to draw some preliminary conclusions: In universities models were used as teaching aids as testified by the significant presence of collections in various Italian universities (F. & N. Palladino 2009 [9]). The creation of models did not greatly interest Italian mathematicians because of the nature of their research, and no adequate industry, comparable to that in Germany, had developed in Italy at the end of the 1800s. In primary school and in the lower-level secondary schools the presence of carton or wood models of elementary solids was widespread enough (Paravia's, Agnelli's, Vallardi's, Mondadori's collections, exhibitions, etc), even if their use seems often to be static, that is based only on the passive observation, also after the diffusion of the methods of the *école active* in Italy. In upper-level secondary schools it appears (textbooks, legislative measures, collections of models, etc.) that the use of models–for the study of elementary and descriptive geometry and of crystallography–was not infrequent, especially in technical and professional schools and institutes, while in *licei classici* (humanistic secondary schools) a teaching of a theoretical and abstract nature prevailed.

[86]E. Castelnuovo 1948 [14], *Geometria intuitiva per le scuole medie inferiori*, Roma: Carabba, p. 160.

[87]E. Castelnuovo 1963, *Didattica della matematica*, Firenze: La Nuova Italia, pp. 105-109.

[88]For more on this subject and for the connections with the international research see L. Giacardi, R. Zan (eds.) 2013, Emma Castelnuovo. L'insegnamento come passione, *La Matematica nella Società e nella Cultura*, s.I, VI, pp. 1-193.

As the use of artefacts became accepted practice in mathematics teaching in Italy (Emma Castelnuovo) and in other countries (Gattegno, Châtelet, Piaget, etc.) in pre-university schools, models, in the broad sense of the term, acquired new importance. That importance has remained today, even though dynamic geometry software has partly replaced physical models with virtual models that are more ductile. However, the problem of whether the processes activated by the manipulation of software are the same as those activated by the manipulation of physical models is still open[89].

Acknowledgments

I am very grateful to François Apéry, Mariolina Bartolini Bussi, Claude Bruter, Renzo Caddeo, Silvia Capuzzo, Giorgio Chiosso, Erika Luciano, Nicla Palladino, Chiara Pizzarelli, David Rowe, Simonetta Di Sieno, Rossana Tazzioli, Carla Vittone for their help or suggestions. A special thanks to Roberto Scoth for his aid in research concerning school legislation and Pompeo Vagliani for information on the catalogues of school materials.

References

On Models in Italy

[1] F. Palladino 1991, Sui modelli matematici in gesso del Dipartimento di Matematica dell'Università di Messina, *Rendiconti del Seminario di Matematica dell'Università di Messina*, Serie II, vol. I, 151-158.

[2] F. Palladino 1992, Antichi strumenti e modelli matematici conservati a Napoli e a Pisa, *Physis*, vol. XXIX (1992) Nuova Serie, 833-847.

[3] F. Palladino e G. Ferrarese 1998, Sulle collezioni di modelli matematici dei Dipartimenti di Matematica dell'Università e del Politecnico di Torino, *Nuncius* (Istituto e Museo di Storia della Scienza di Firenze), XIII, 169-185.

[4] L. Carbone, G. Cardone e F. Palladino 1998, Le collezioni di strumenti e modelli matematici del Dipartimento di Matematica e Applicazioni "R. Caccioppoli" dell'Università "Federico II" di Napoli. Cataloghi ragionati, *Rendiconto dell'Accademia delle Scienze Fisiche e Matematiche di Napoli*, (IV), vol. LXV, anno CXXXVII, 93-257.

[5] F. Palladino 1999, *Il Fondo di Strumenti e Modelli matematici antichi dell'Università di Padova e l'iniziativa di Giuseppe Veronese per un Laboratorio Nazionale Italiano*, Padova: Università di Padova - Dipartimento di Matematica pura ed applicata.

[6] F. Palladino & N. Palladino 2001, Sulle raccolte museali italiane di modelli per le matematiche superiori. Catalogo ragionato e sito web, *Nuncius* (Istituto e Museo di Storia della Scienza di Firenze), XVI, 781-790.

[89] On mathematical concrete and virtual manipulatives, see M. G. Bartolini Bussi, F. Martignone, Manipulatives in Mathematics Education, in Lerman, Steve (ed.), *Encyclopedia of Mathematics Education* Berlin: Springer, 2014, pp. 365-372.

[7] L. GIACARDI 2003, La collezione di modelli geometrici della Biblioteca speciale di matematica "G. Peano", in G. Giacobini (ed.), *La memoria della scienza. Musei e collezioni dell'Università di Torino*, Torino: Fondazione CRT, 251-256.

[8] G. FERRARESE (ed.) 2004, *La collezione dei modelli geometrici della Biblioteca di Matematica "G. Peano"*, Torino: Dipartimento di Matematica, 2004.

[9] N. & F. PALLADINO 2009, Le collezioni museali del Dipartimento di Matematica e Applicazioni "R. Caccioppoli" dell'Università di Napoli "Federico II", in T. Bruno (ed.) Atti del Convegno in onore di Carlo Ciliberto, *Ratio Mathematica*, 19, 31-88.

Other references

[10] B. BELHOSTE, H. GISPERT & N. HULIN (eds.) 1996, *Les sciences au lycée. Un siècle de réformes des mathématiques et de la physique en France et à l'étranger*, Paris: Vuibert-INRP.

[11] L. BOI, L. GIACARDI & R. TAZZIOLI 1998, *La découverte de la géométrie non euclidienne sur la pseudosphère. Les lettres de Beltrami à Hoüel, avec une introduction des notes et commentaires critiques*, Paris: Blanchard.

[12] C. CAPELO & M. FERRARI 1982, La cuffia di Beltrami: storia e descrizione, *Bollettino di storia delle scienze matematiche* 2, 233-247.

[13] G.CARDONE 1996, Il Congresso della Mathesis del 1921: R. Marcolongo ed i modelli matematici, in *Mathesis centenario 1895-1995: Una presenza nella cultura e nell'insegnamento*, Roma, 147-152.

[14] G. CASTELNUOVO 1928, La geometria algebrica e la scuola italiana, *Atti del Congresso Internazionale dei Matematici*, Bologna: Zanichelli, vol. I, 191-201.

[15] M. BARTOLINI BUSSI, D. TAIMINA, & M. ISODA 2010, Concrete models and dynamic instruments as early technology tools in classrooms at the dawn of ICMI: from Felix Klein to present applications in mathematics classrooms in different parts of the world. *ZDM-The International Journal on Mathematics Education*, 42(1), 9-31.

[16] A. BRIGAGLIA, C. CILIBERTO 1995, *Italian Algebraic Geometry between the two World Wars*, Queen's Papers in Pure and Applied Mathematics, vol. 100, Queen's University.

[17] A. BRIGAGLIA 2006, Da Cremona a Castelnuovo continuità e discontinuità nella visione della scuola, in L. Giacardi, (ed.) 2006. *Da Casati a Gentile. Momenti di storia dell'insegnamento secondario della matematica in Italia*, Pubblicazioni del Centro Studi Enriques, La Spezia: Agorà Edizioni.

[18] F. ENRIQUES, *Per la storia della logica*, Bologna: Zanichelli, 1922.

[19] G. FISCHER 1986, *Mathematical Models from the Collections of Universities and Museums*, Braunschweig: F. Vieweg & Sohn.

[20] L. GIACARDI 2003a, Educare alla scoperta. Le lezioni di Corrado Segre alla Scuola di Magistero, *Bollettino dell'Unione Matematica Italiana. La Matematica nella società e nella cultura* (8), VI-A, 2003, 141-164.

[21] L. GIACARDI 2009, The School as a "Laboratory." Giovanni Vailati and the Project for the Reform of the Teaching of Mathematics in Italy, *International Journal for the History of Mathematics Education*, 4.1, 5-280.

[22] L. GIACARDI 2011, The Emergence of the Idea of the Mathematics Laboratory in the Early Twentieth Century, in K. Bjarnadóttir et al., *Proceedings of the Second International Conference on the History of Mathematics Education*, Lisbona: UIED, 203-225.

[23] H. GISPERT 2002, Introduction au texte: E. Borel 1904, Les exercices pratiques de mathématiques dans l'enseignement secondaire, SMF, Gazette, 93.
http://smf4.emath.fr/Publications/Gazette/2002/93/smf_gazette_93_47-64.pdf.

[24] E. LUCIANO, C. S. ROERO, C.S. 2012, From Turin to Göttingen: dialogues and correspondence (1879-1923), *Bollettino di Storia delle Scienze Matematiche*, XXXII, 9-232.

[25] R. MARCOLONGO 1922, Materiale didattico di Battaglini, *Giornale di matematiche*, XL, 1-14.

[26] R. MARCOLONGO 1935, *Quaranta anni di insegnamento*, Napoli: SIEM.

[27] M. MENGHINI (ed.) 1994, La corrispondenza di Luigi Cremona (1830-1903), *Quaderni della Rivista di Storia della Scienza*, 3, vol. II.

[28] M. MENGHINI (ed.) 1996, Per l'Archivio della corrispondenza dei Matematici italiani. La corrispondenza di Luigi Cremona (1830-1903), vol. III, *Quaderni P.RI.ST.EM.*, Palermo, Milano: Università Bocconi.

[29] A. MILLÁN GASCA (ed) 1992, La corrispondenza di Luigi Cremona (1830-1903), *Quaderni della Rivista di Storia della Scienza*, 1, vol. I.

[30] Modellaustellung, in *Verhandlungen des dritten internationalen Mathematiker-Kongresses in Heidelberg vom 8 bis 13 August 1904*, Leipzig: G. B. Teubner, 1905, 731-736.

[31] Modelli per gli insegnamenti di Geometria e di Analisi, *Bollettino della Unione Matematica Italiana*, 3, VI, 1951, 366.

[32] L. NURZIA (ed.) 1999, Per l'Archivio della corrispondenza dei Matematici italiani. La corrispondenza di Luigi Cremona (1830-1903), vol. IV, *Quaderni P.RI.ST.EM.*, Palermo, Milano: Università Bocconi.

[33] K. H. PARSHALL, D. E. ROWE 1994, *The emergence of the American Mathematical research community, 1876-1900: J.J. Sylvester, F. Klein and E.H. Moore*, AMS, 1994.

[34] C. PIZZARELLI, L'istruzione matematica secondaria e tecnica da Boncompagni a Casati 1848-1859: il ruolo della Società d'Istruzione e di Educazione, *Rivista di Storia dell'Università di Torino*, 2, II, 2013, pp. 23-60.

[35] D. ROWE & J. MCCLEARY (eds) 1989, *The History of Modern Mathematics*, London: Academic Press, vols. I, II.

[36] D. ROWE 2013, Mathematical Models as Artefacts for Research: Felix Klein and the Case of Kummer Surfaces, *Mathematische Semesterberichte*, 60.1, 1-24

[37] C. SEGRE 1891, Su alcuni indirizzi nelle investigazioni geometriche. Osservazioni dirette ai miei studenti, *Rivista di Matematica*, 1, 42-66; Opere, 4, 387-412.

[38] G. SCHUBRING 2010, Historical comments on the use of technology and devices in ICMEs and ICMI. *ZDM -The International Journal on Mathematics Education*, 42 (1), 5-9.

[39] A. TERRACINI 1968, *Ricordi di un matematico. Un sessantennio di vita universitaria*, Roma: Ed. Cremonese

[40] O. ZARISKI 1943, Review of: B. Segre, The Non-singular Cubic Surfaces. Oxford University Press, Oxford, 1942. In *Mathematical Reviews* 1943.

Websites of interest

www.dma.unina.it/ nicla.palladino/catalogo
www.dm.unito.it/modelli/index.html
www.luigi-cremona.it.

Livia Giacardi
Professor (Università degli Studi di Torino)
Dipartimento di Matematica
Via Carlo Alberto 10
10123 Torino
livia.giacardi@unito.it

CARON'S WOODEN MATHEMATICAL OBJECTS

François Apéry

Abstract

We pay tribute to the mathematician-artist Joseph Caron by showing his collection of wooden models stored in the Institute Henri Poincaré in Paris.

1 The mathematician

The collection of mathematical objects of the Institute Henri Poincaré in Paris includes in particular a series of wooden models made by Joseph Caron (Fig. 1). Joseph Caron entered the École Normale Supérieure in Paris in 1868. He was appointed professor of descriptive geometry at several parisian Lycées starting from 1871. The next year, according to the wish of Gaston Darboux, he was designated as director of graphical works at École Normale Supérieure. He wrote several handbooks of descriptive geometry ([2],[3],[4]). His acute sense of geometric reality, combined with an interest in practical realizations, stimulated the students amongst them the young Henri Lebesgue whose taste turned therefore towards geometrical constructions. Actually, Caron produced physical models of geometrical objects stemming from the exercices and lectures by Darboux on curves and surfaces (see [5]), and so doing, he supplied the Cabinet de mathématiques (a kind of Cabinet of curiosities for mathematical beings directed by Darboux) at the Sorbonne (Fig. 2).

From 1872 to January 1915 (the first World War possibly stopped the work) he produced more than hundred models mainly in wood. Most of them are signed, however some of the first ones are not, maybe because nobody made him aware of the future history of science. For instance, in 1880 he wrote an article entitled *Sur l'épure des 27 droites d'une surface du troisième degré dans le cas où ses droites sont réelles* in the Bulletin de la Société Mathématique de France, and he is likely (this is a conjecture) the author of the model herebelow (Fig. 3) (Man Ray took a picture of it in the 30's). In this paper, we shall focus on wooden models, and particularly on a series of eight models illustrating a point in optical theory.

Fig. 1 Joseph Caron 1849-1924

Fig. 2 Cabinet de mathématiques at the Sorbonne

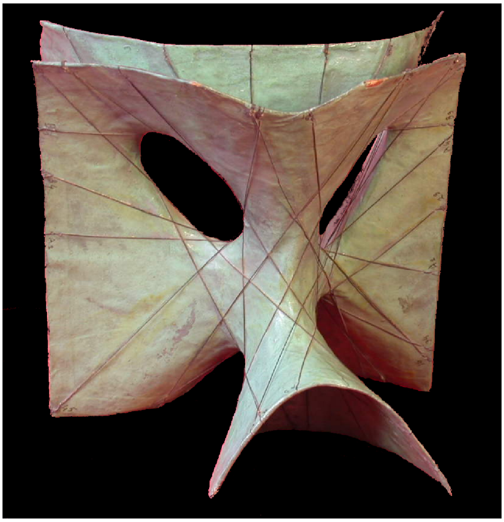


Fig. 3 Cubic surface with 27 real lines

2 Selection of wooden models

Here below is a selection of wooden models made by Caron between 1910 and 1915 (Fig. 4, 5, 6, 7, 8, 9, 10).

Fig. 4 From left to right: Rational algebraic surface of degree eight generated by the plane section of a cylinder rolling on another cylinder. Rational algebraic surface of degree ten. Envelope of the normals for a hyperbolic paraboloid.

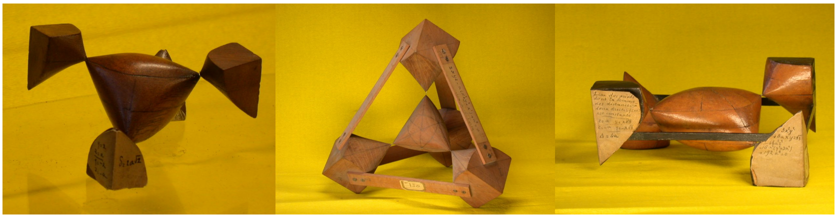


Fig. 5 From left to right: Algebraic surface of degree four defined by the set of points whose sum of the distances to two lines is constant. Algebraic surface of degree three with tetrahedral symmetry. Algebraic surface of degree four defined by the set of points whose sum of the distances to two lines is constant.

Fig. 6 Three algebraic surfaces of degree four.

Fig. 7 From left to right: Envelope of the normals for a Plücker conoid. Algebraic surface of degree four. Kummer surface with twelve real double points.

Fig. 8 From left to right: Set of three deformations of an ellipsoid. Two spiral surfaces generated by circles.

Fig. 9 From left to right: Henneberg's minimal surface. Rational surface of degree four.

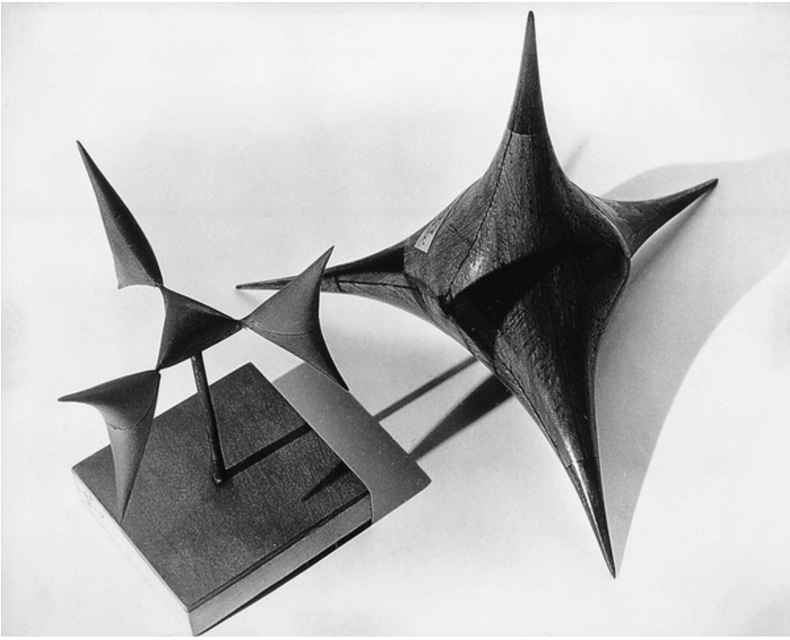


Fig. 10 Two algebraic surfaces of degree four

Figure 10 requires an explanation. The left-hand model is the same as the right one shown on figure 6. The point is that the right-hand model has been lost. Thanks to Man Ray who photographed both models in 1935, we have a record of its existence.

3 A series of eight models made between 1912 and 1914.

The problem stated by Darboux is as follows: find the surfaces orthogonal to the lines three points of which of mutually constant distances moving on three orthogonal planes (Fig. 11).

It takes its origin in optical theory: find a new geometric definition of a wave front following the work of Malus, Dupin, Niven,.... A wave front is the surface generated at the time t fixed, by the electromagnetic particles emitted by a body at time t_0. Therefore it is orthogonal to the rays of particles. Such a wave front generates a caustic where the energy is concentrated. It is a surface tangent to all the rays of particles. The eight Caron's models below illustrate these notions (Fig. 12).

While time is varying, the shape of the wave front changes smoothly, but sometimes singularities pop up, that is points where the curvature is no longer

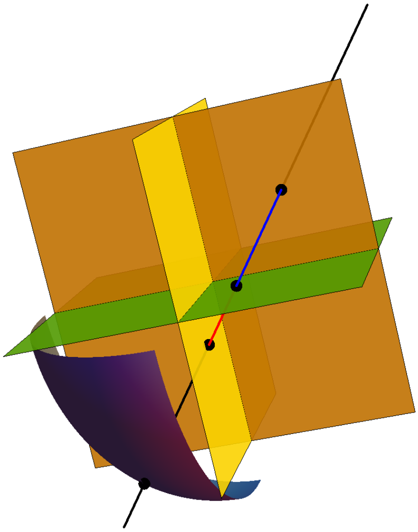


Fig. 11 The surface (violet) is orthogonal to the moving line. The three orthogonal planes cut two segments (blue and red) of constant lengths on the line).

Fig. 12 (three left-hand columns) Wave front at six different values of the time. (rightmost column) Caustics.

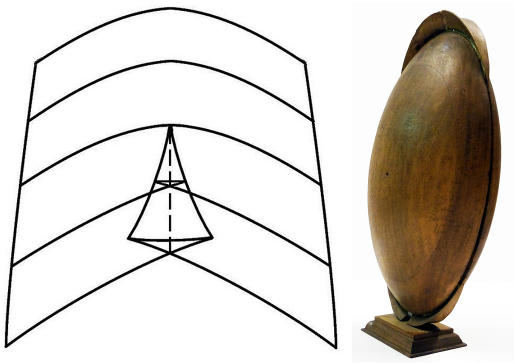


Fig. 13 (*left*) Swallowtail at the cusp of a cuspidal edge. (*right*) Four swallowtails on the Caron's model.

bounded, like on a cuspidal edge or a swallowtail (Fig. 13). At certain particular values of the time, a singularity not existing before, suddenly appears. The shape of the wave front changes drastically. Such a phenomenon is called a bifurcation or a metamorphosis. Between two metamorphoses, either the global shape remains smoothly stable, like a sphere deformed by small hollows or bumps, or self-intersection occurs or is recomposed by surgery. The key point to understand the evolution of the wave front consists therefore in describing self-intersections and metamorphoses. In a generic wave front, the metamorphoses have been classified by V.I. Arnol'd in 1974 ([1]). There are five types shown below (Fig. 14).

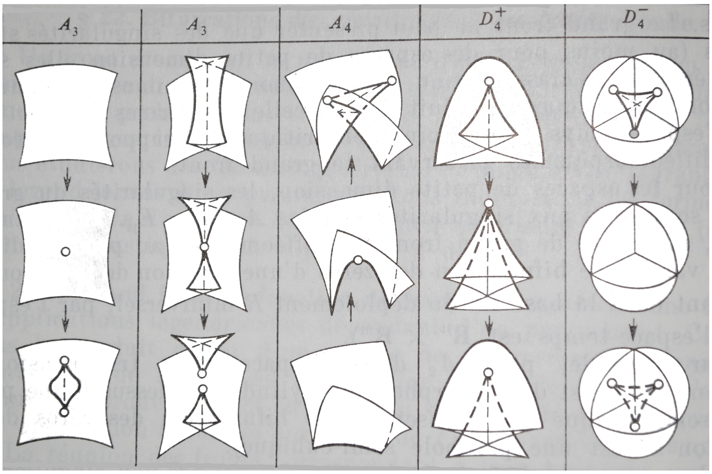


Fig. 14 (*left*) The five metamorphoses of a generic one-parameter wave front.

It is noticeable that the Caron's models are precisely chosen in order to suggest the transitions between nonequivalent shapes as shown in figures 15, 16. Similarly, looking at other pairings of models, we can recognize a quadruple point, the birth of a self-intersection curve, and the birth of two swallowtails (type A_3).

Fig. 15 Swallowtails hyperbolic confluence (type A_3)

Fig. 16 Double swallowtail (type D_4^+)

4 The artist

The surrealists never mentionned the name of Caron, they gave him no credit for his work even though most of the models they were using were signed. Of course Caron was dead, but today we can't imagine applying such a treatment to Dali, Ernst, Man Ray and others. Maybe, Caron did not consider himself as an artist, however his wooden models are nicely finished, the wood is polished and varnished, the models are attached on structures to be presented on supports. And last but not least, the surrealists touch them and insert them in their own productions. That is a kind of definition of a piece of art. As a consequence we should consider Joseph Caron as a mathematician-artist.

References

[1] V.I. Arnol'd, *Méthodes mathématiques de la mécanique classique*, Mir 1976 (traduction de l'édition russe de 1974).

[2] J. Caron, *Cours de géométrie descriptive (droite et plan). À l'usage des aspirants au baccalauréat ès-sciences*, librairie Germer Baillière et Cie, Paris 1882.

[3] J. Caron, *Cours de géométrie descriptive. À l'usage des classes de mathématiques spéciales*, librairie E. Foucart, Paris 1883.

[4] J. Caron, *Cours de géométrie descriptive. Géométrie cotée. À l'usage des candidats à l'École Spéciale Militaire de Saint-Cyr*, librairie Germer Baillière et Cie, Paris 1898.

[5] G. Darboux, *Leçons sur la théorie générale des surfaces*, première partie, Gauthier-Villars, Paris (1914).

Illustration Credits

Fig. 1. Courtesy of the École Normale Supérieure library.
Fig. 2. Courtesy of the Institut Henri Poincaré.
Fig. 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 15, 16. Models pertaining to the Institut Henri Poincaré collection. Photographs by the author.
Fig. 10. After a photography by Man Ray, Cahiers d'Art, n° 1-2, 1936.
Fig. 14. Illustration from [1].

François Apéry
Associate Professor (University of Mulhouse)
Scientific referee for the collection of mathematical models
at the Institut Henri Poincaré
ESMA, c/o Institut Henri Poincaré
11 rue P. & M. Curie
75321 Paris Cedex 05
francois.apery@uha.fr

POLYHEDRAL EVERSIONS OF THE SPHERE. FIRST HANDMADE MODELS AND JavaView APPLETS

Richard Denner

Abstract

This article gives the tools for self-construction of the polyhedral models which appear during the process of everting the polyhedral sphere. It can be understood as a pedagogical device to understand the different steps of that process.

1 Introduction

There are problems which are real challenges. The sphere eversion problem belongs to that category! How is it possible to exchange the internal and the external face of a sphere without tearing or folding its surface? At first sight, it seems impossible! But, by authorizing the surface to cross itself and by respecting rules established by mathematicians, it becomes possible and the eversion[1] can be shown on the screen of a computer. At the conference, I presented pictures of polyhedra imagined by the blind mathematician Bernard Morin which illustrate the sphere eversion. The approach which is developed here allowed the discovery of the first *eversion of the cuboctahedron*. We present three introductory models which lead directly to the central stage of the eversion.

The starting point of this collection of models is a minimal Boy surface with 9 vertices inspired by Ulrich Brehm's work [1]. It is a non-orientable surface which presents a threefold axis of symmetry. Some of its faces intersect themselves giving birth to an intersection line and a *triple point*. The same construction process can be applied to get a model with a fourfold symmetry called *open halfway-model*. Then the surface becomes orientable and has a *quadruple point*. The reader is invited to build by himself these two first models. The third model, called *closed halfway-model*, reaches the necessary level of complexity to carry out the eversion of the cuboctahedron. Handmade models, photos in artificial light and JavaView[2] applets were used to highlight the thought of the blind mathematician.

[1] http://torus.math.uiuc.edu/optiverse/
[2] http://www.javaview.de/

For the exchanges with Morin, we used a closed halfway-model in white drawing paper, less aesthetic than the one presented here. Impossible to see inside! The construction had to be improved with transparent faces in rhodoïd and with the use of two different colours (red and blue) for each side of the opaque faces. Many years later, the models were realized on computer with Konrad Polthier's software JavaView; all the details of the eversion can be shown. The two sides of a face can be displayed with two different colours as on the handmade models. Furthermore, JavaView is able to handle the intersections; the triple point and the quadruple point are immediately visualized as intersections of three or four faces.

2 Minimal polyhedral Boy surface

At the beginning of the 1980's, during summer holidays readings I fell casually on blind mathematician Bernard Morin's article "Le retournement de la sphère" [2] illustrated by Jean-Pierre Petit's drawings in the revue Science. It aroused my curiosity and I tried to understand step-by-step the sphere eversion[3] that Morin had imagined. A few years later (1986), I met him at the University Louis Pasteur of Strasbourg. At the end of the formation he invited me, with other colleagues, in his office to show us a great wire model of the Boy surface ([3], [4]). I immediately recognized – and was fascinated by – the surface I discovered a few years earlier in his article. After our discussion, he gave me a letter written by Ulrich Brehm which contained a short description of a Boy surface with nine vertices. It was a variant of the minimal Boy surface conceived by Brehm [1] which Morin has adapted to the polyhedral sphere eversion. I tried to build it briefly and succeeded after a few days.

2.1 Construction of a polyhedral Boy surface

Boy surfaces[4] are obtained by gluing together a Möbius band[5] and a disk along their boundaries. The first model we will describe is Ulrich Brehm's polyhedron[6]. Its Möbius band is a three half-turns twisted band; it is a remarkable assembly of three concave pentagons which is explained below.

The first pentagon $P_0 = C_0A_0B_0A_1B_1$ respects the following conditions :

1. the triangle $A_0B_1C_0$ is equilateral,
2. the point B_0 is its orthocenter,
3. the vertex A_1 is so that the quadrangle $C_0B_0A_1B_1$ is a parallelogram.

[3] http://www.lutecium.org/jp-petit/science/maths_f/Retournement_sphere/PLS_79.pdf
[4] http://arpam.free.fr/The%20Boy%20Surface%20as%20Architecture%20and%20Sculpture.pdf
[5] http://www.mathcurve.com/surfaces/mobius/mobius.shtml
[6] http://www.mathcurve.com/polyedres/brehm/brehm.shtml

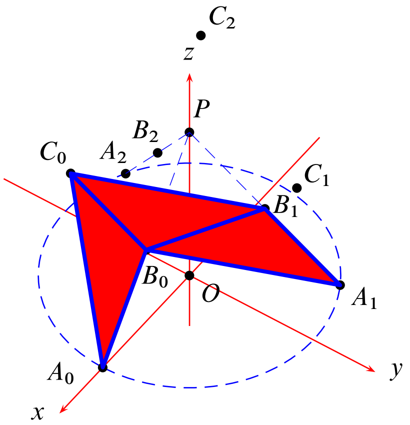


Fig. 1 The pentagonal face P_0 and its triangulation. Note that the three triangles in red are isosceles and have an angle which measure is 120°. Simple and nice! During the deformation the pentagons can be folded along the sides of the triangles. This picture is realized with LaTeX and the packages pstricks and pst-3dplot.

Similarly, we construct two other pentagons $P_1 = C_1A_1B_1A_2B_2$ and $P_2 = C_2A_2B_2A_0B_0$. These 3 pentagons lean towards the faces of a regular tetrahedron $PA_0A_1A_2$.

The coordinates of the nine vertices are:

$$
\begin{array}{lll}
A_0(1;0;0) & A_1(-\frac{1}{2};\frac{\sqrt{3}}{2};0) & A_2(-\frac{1}{2};-\frac{\sqrt{3}}{2};0) \\
B_0(\frac{1}{2};0;\frac{\sqrt{2}}{2}) & B_1(-\frac{1}{4};\frac{\sqrt{3}}{4};\frac{\sqrt{2}}{2}) & B_2(-\frac{1}{4};-\frac{\sqrt{3}}{4};\frac{\sqrt{2}}{2}) \\
C_0(\frac{3}{4};-\frac{\sqrt{3}}{4};\sqrt{2}) & C_1(0;\frac{\sqrt{3}}{2};\sqrt{2}) & C_2(-\frac{3}{4};-\frac{\sqrt{3}}{4};\sqrt{2})
\end{array}
$$

If we consider the assembly $P_0 \cup P_1 \cup P_2$ we get a polyhedral Möbius band. We just have to add 7 triangles which assembly is homeomorphic to a disk (see Fig. 2-b):

- three dorsal triangular faces $Q_0 = C_0B_1A_2$, $Q_1 = C_1B_2A_0$ and $Q_2 = C_2B_0A_1$; their intersection is the triple point,
- three ventral triangular faces $R_0 = C_0A_2A_0$, $R_1 = C_1A_0A_1$ and $R_2 = C_2A_1A_2$,
- and to finish the equilateral triangle $A_0A_1A_2$.

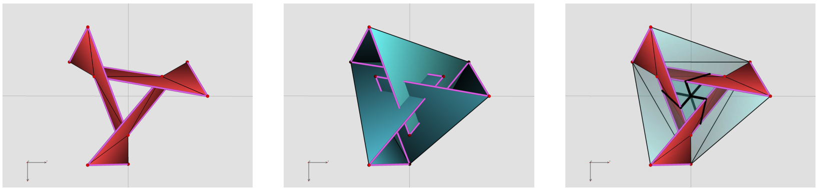


Fig. 2 Construction of the Boy surface by gluing together a Möbius strip and a disk.

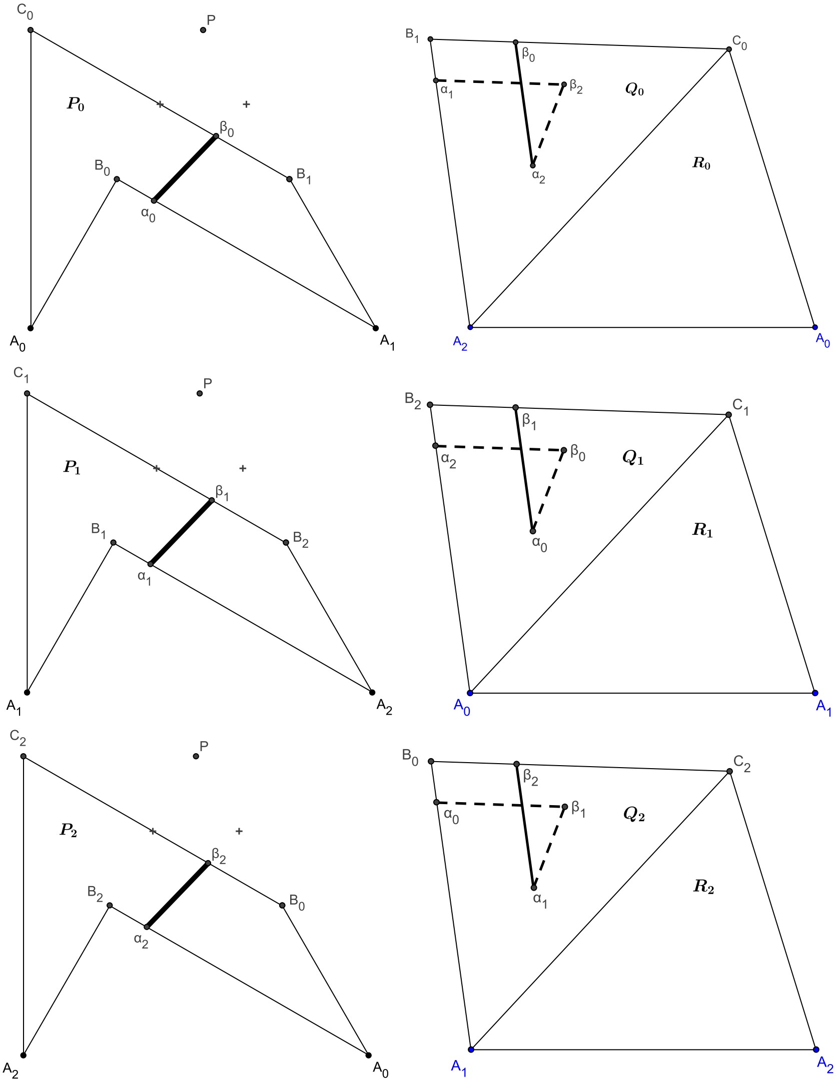


Fig. 3 Assembly of the Boy surface; scale = 0.4. Notation: $\alpha_0 = P_0 \cap Q_2 \cap Q_1$, $\beta_0 = P_0 \cap Q_0 \cap Q_1$. Begin to bring together Q_2 and Q_1, along their intersection line $[\alpha_0\beta_1]$; then insert Q_0 into the previous assembly. The trick is easy and you will succeed quickly. Now add the three pentagons by pushing them through their corresponding slots $[\alpha_0\beta_0]$, $[\alpha_1\beta_1]$ and $[\alpha_2\beta_2]$ on the dorsal faces: the Möbius strip takes its place. Then, to finish add the ventral faces R_0, R_1 and R_2; and if you want to close the model add a last equilateral triangle $A_0A_1A_2$ – which has the same size as the triangle PA_0A_1 – as bottom face. Enjoy! An **important fact** to notice here is that the **flexibility** of the material (paper or rhodoïd) is very useful for the assembly.

A nice description of the construction of the Boy surface and more topological reminders are available in Laura Gay's internship report[7] at the Institute Camille Jordan in Lyon. See also [6] for Boy surfaces having a higher level of symmetry.

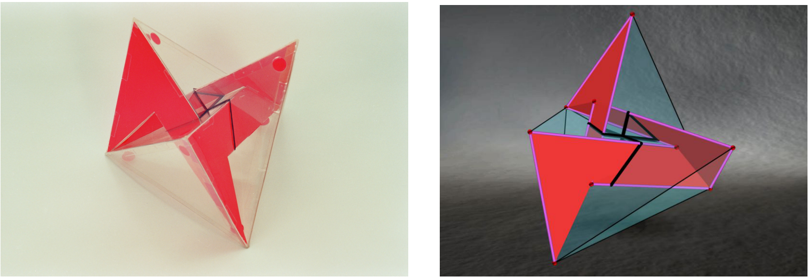


Fig. 4 Minimal Boy surface with 9 vertices: handmade model and JavaView applet.

3 Open halfway-model

With 4 concave pentagons in vertical position we get a model with 12 vertices:

$A_0(3;-3;0)$	$A_1(3;3;0)$	$A_2(-3;3;0)$	$A_3(-3;-3;0)$
$B_0(3;-3;6)$	$B_1(3;3;6)$	$B_2(-3;3;6)$	$B_3(-3;-3;6)$
$C_0(3;-15;8))$	$C_1(15;3;8)$	$C_2(-3;15;8)$	$C_3(-15;-3;8)$

Its 12 faces are $P_i = C_i A_i B_i A_{i+1} B_{i+1}$, $Q_i = C_i B_{i+1} A_{i+2}$ and $R_i = C_i A_{i+2} A_i$ where $i \in \mathbb{Z}/4\mathbb{Z}$.

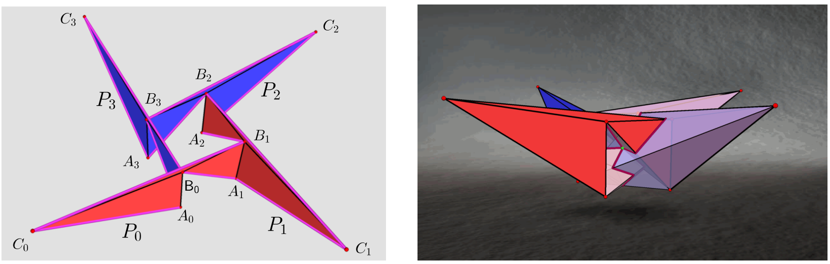


Fig. 5 Open halfway-model: the quadruple point is reachable by passing under the pentagons.

[7] http://math.univ-lyon1.fr/~borrelli/Jeunes/rapport_de_stage_Laura_Gay.pdf

3.1 Intersection line

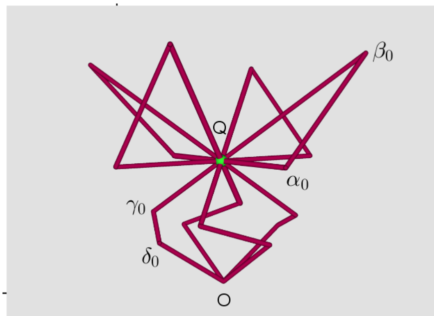


Fig. 6 For the construction of the model – and the JavaView applet – all the coordinates of the points which determine the self-intersection line had to be calculated by solving several linear systems. Here we have annotated $\alpha_0 = P_0 \cap Q_3 \cap Q_1$, $\beta_0 = P_0 \cap Q_0 \cap Q_1$, $\gamma_0 = Q_0 \cap R_0 \cap Q_1$ and $\delta_0 = Q_0 \cap Q_1 \cap R_1$. They all belong to the plane Q_1 like the quadruple point Q (in green).

3.2 Construction of an open halfway-model

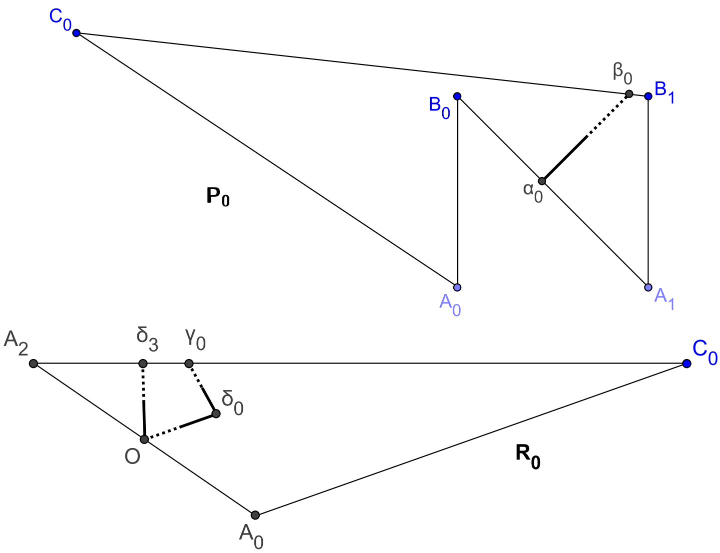


Fig. 7 The pentagonal face P_0 and the ventral face R_0; scale = 0.5. They can be used as template for the other faces P_i and R_i, $i = 1,...,3$. The geometrical figures are reproduced with GeoGebra.

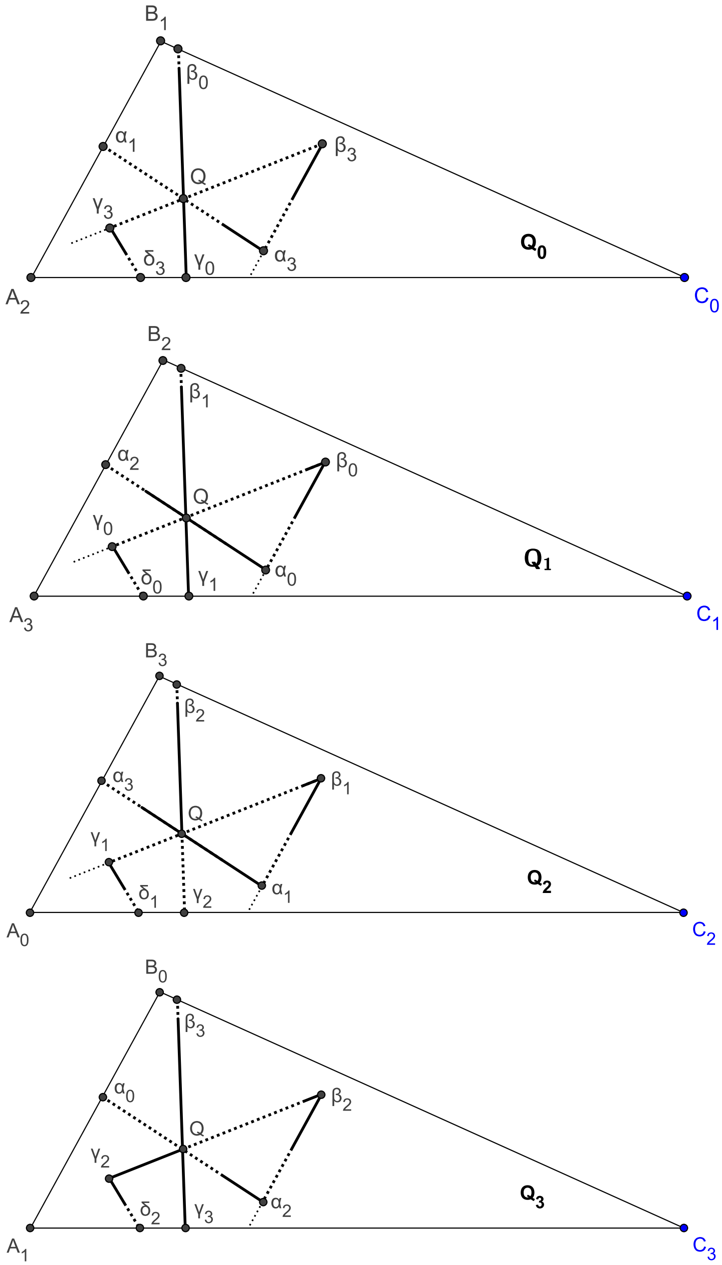


Fig. 8 The four dorsal faces Q_0, Q_1, Q_2 and Q_3; scale = 0.5. They have in common the quadruple point Q. It was crucial to find how to do this assembly.

3.3 Tips for the mounting of the model

Don't use simple paper, it will not work easily. Use at least drawing paper which has a better rigidity.

Fig. 9 Open halfway model: handmade models need ability, precision and perseverance. The pentagons are obtained by gluing together two cardboard sheets – one side in red for the one and one side in blue for the other. To work the rhodoïd, a steel edge and a fine cutter are necessary. To mark the rhodoïd from the plans, needles and a small hammer were used. Faces are fixed together with adhesive tape. The self-intersection line is drawn by using a pencil with permanent ink.

1. Take Q_2 in your left hand.
2. Take Q_1 in your right hand, then push Q_1 into the slot $[\gamma_1\beta_1]$ of Q_2 until the points γ_1 and β_1 of the two faces are touching each other.
3. Now, take Q_0 in your right hand. Try to insert Q_0 into the slot $[\gamma_0\beta_0]$ of Q_1 and at the same time join Q_2 and Q_0 along the segment $[\alpha_1\alpha_3]$.
4. To finish the assembly of the quadruple point, take Q_3 in your right hand. The goal is to push Q_3 through the slot $[\gamma_2\beta_3]$ of Q_0 and at **the same time** Q_1 and Q_3 have to be joined along the segment $[\alpha_0\alpha_2]$. Moreover, Q_2 and Q_3 have to be joined along the segment $[\gamma_2\beta_2]$!
   There is a trick to do this! The flexibility of the matter here is absolutely necessary.
   The trick consists with your left hand to flatten together Q_2 and Q_1 between your thumb and your index finger – level with point β_2 – so that they can be pushed together into the slot $[\alpha_0 Q]$ of Q_3 until they reach the quadruple point Q on Q_3. Then β_2 on Q_2 can move towards β_2 on Q_3. α_2 on Q_1 can move towards α_2 on Q_3. β_3 on Q_3 can move towards β_3

on Q_0. The quadruple point Q can now be assembled by pushing all the points in their right position.
5. Add the four pentagons.
6. Add the four ventral faces.

4 Closed halfway-model of the eversion of the cuboctahedron

This model is better suited to realize the eversion than the previous model.

4.1 Description of the construction

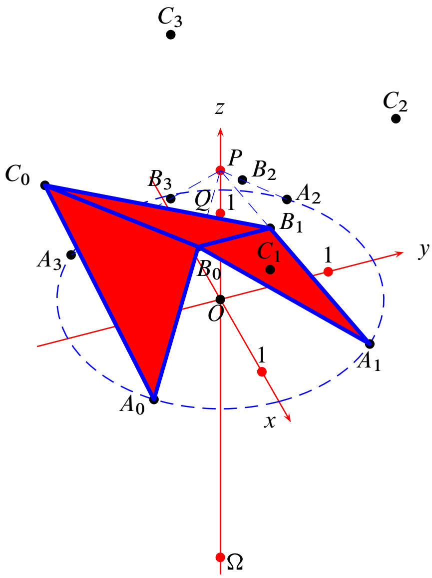


Fig. 10 The pentagon P_0 at the central stage; remember that $Q_0 = C_0B_1A_2$ and $R_0 = C_0A_2A_0$.

On this third model,
1. The four pentagons lean against the lateral faces of the regular pyramid $PA_0A_1A_2A_3$ where the basis is determined by the vertices $A_0(1;-1;0)$, $A_1(1;1;0)$, $A_2(1;-1;0)$ and $A_3(-1;-1;0)$ and where the apex is $P(0;0;\frac{3}{2})$.
2. $B_i \in [PA_i]$ and their third coordinate is 1; furthermore $\boxed{B_i \in Q_{i+1}}$ for $i \in \mathbb{Z}/4\mathbb{Z}$. Consequently, the accesses to the quadruple point Q are closed by the dorsal faces: the halfway-model is said "closed".

3. Let Ω be the point $\Omega(0;0;-3)$ and V_3 the plane $(A_3\Omega A_0)$. Then the coordinates of the vertex C_0 result from $C_0 = P_0 \cap (A_2B_3B_1) \cap V_3$.

All the coordinates can then be calculated. The quadruple point is the point $Q(0;0;1)$.

4.2 Coordinates

$A_0(1;-1;0)$	$A_1(1;1;0)$	$A_2(-1;1;0)$	$A_3(-1;-1;0)$
$B_0(\frac{1}{3};-\frac{1}{3};1)$	$B_1(\frac{1}{3};\frac{1}{3};1)$	$B_2(-\frac{1}{3};\frac{1}{3};1)$	$B_3(-\frac{1}{3};-\frac{1}{3};1)$
$C_0(-\frac{1}{7};-\frac{11}{7};\frac{12}{7})$	$C_1(\frac{11}{7};-\frac{1}{7};\frac{12}{7})$	$C_2(\frac{1}{7};\frac{11}{7};\frac{12}{7})$	$C_3(-\frac{11}{7};\frac{1}{7};\frac{12}{7})$

4.3 Decomposition in several geometries with JavaView

Inside a JavaView applet, it is possible to create different *geometries*; the following pictures illustrate this possibility. Pentagonal faces, dorsal faces, and ventral faces are represented separately for a better understanding of the model. The self-intersection line has also been added after calculation of all the vertices. We touch here the limits of the software: it could be very useful to have a software which allows to isolate directly the self-intersection line, specially

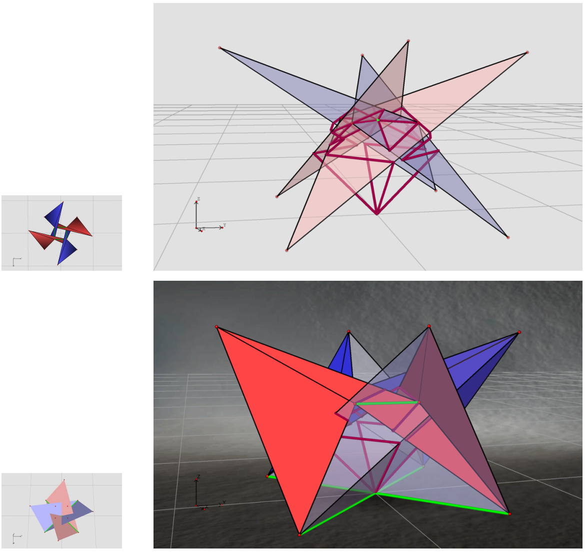


Fig. 11 Closed halfway-model; decomposition in different geometries with JavaView. It is useful to locate the two perpendicular edges $[A_0A_2]$ and$[A_1A_3]$ and the square $B_0B_1B_2B_3$ (in green).

for the study of its evolution along the eversion. This stays actually out of reach with JavaView. The next picture shows the same handmade model photographed in artificial light; the internal subdivision is completely observable. Just under the quadruple point there is a chamber – completely closed towards the outside – which has the shape of an octahedron with four ex-growths like four small teeth. It will be interesting to follow its evolution during the eversion. The second image represents this internal room with the self-intersection line and the quadruple point.

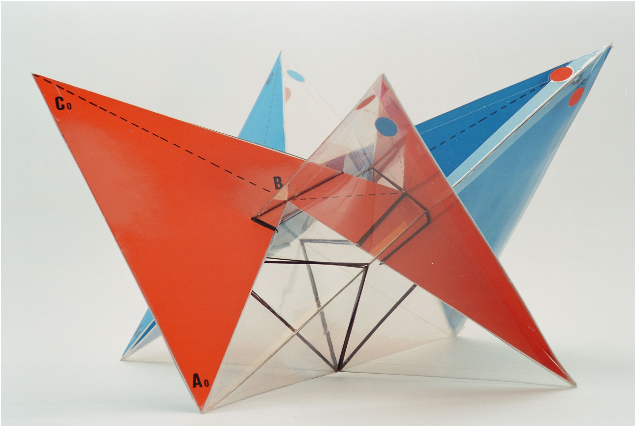


Fig. 12 Closed halfway-model of the eversion of the cuboctahedron. Imagined by Bernard Morin, this model is really the cornerstone of this study.

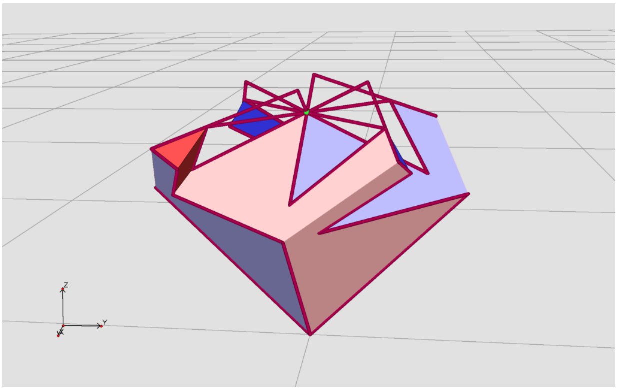
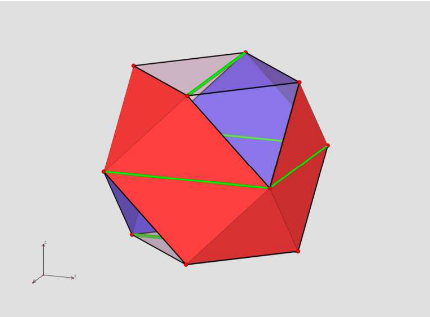


Fig. 13 Internal room under the quadruple point and self-intersection line of the closed halfway-model

See also [7] for halfway models with a higher level of symmetry; the article is illustrated with engravings by Patrice Jeener. The display of the open and the closed halfway-models was enhanced after mail exchanges with Jean Constant. He made two artistic pictures[8] with the use of these models. Enjoy!

5 First eversion of the cuboctahedron

The initial and the final stages of the eversion are obtained by splitting its 6 square faces with 2 orthogonal polar-edges $[A_0A_2]$ and $[A_1A_3]$ and with its equator $B_0B_1B_2B_3$ (in green). By doing so, we get a polyhedron which have exactly the same number of vertices (12), edges (30) and faces (20) as on the triangulated halfway-models!



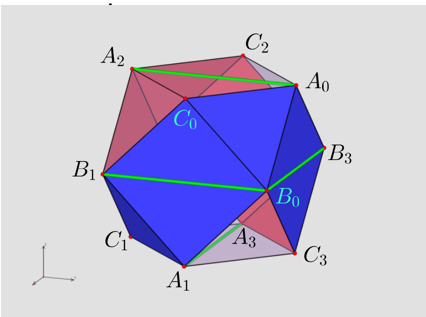


Fig. 14 Initial and final models of the eversion of the cuboctahedron

On the initial model and on the final model, a same vertex has two antipodal positions. Each triangular face is transformed in its antipodal face (see for instance the orientation of the face $A_0C_2B_3$ on the two models), so one can observe that the orientation of the faces has changed on the final model. Similarly, the north polar-edge $[A_1A_3]$ on the first model is changed into the south polar-edge $[A_1A_3]$ on the second model. Observe that these two edges are parallel. The same observation can be done with the south polar-edge $[A_0A_2]$. On the second picture, one can also locate the final position of the pentagon $P_0 = C_oA_0B_0A_1B_1$. Now, the four pentagons of the halfway models are represented by the oscillating belt – composed with 12 triangles – around the equator! The next picture illustrates the problem of the eversion of the cuboctahedron and suggests the question: how does it work?

Bernard Morin conceived a step-by-step deformation which deforms the halfway model by means of *elementary transformations* consisting in moving

[8] http://imaginary.org/fr/node/263

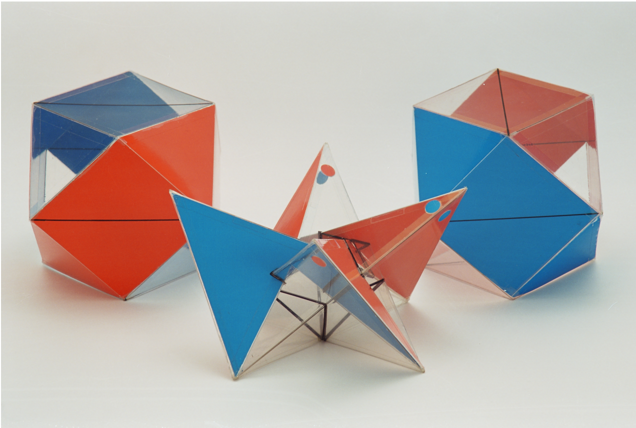


Fig. 15 Initial, halfway and final models of the eversion of the cuboctahedron

a vertex along an edge of the polyhedron. In all, 22 steps are necessary **to transform the halfway model** (step 0) into the final cuboctahedron (in blue). But, **only** 6 steps are needed to obtain a model without self-intersection line! All the models which intervene have a twofold symmetry. What can be done to get the blue cuboctahedron (final step +22) from the halfway model can also be done to get the red cuboctahedron (initial step −22). So, if we consider all the 45 models from the model −22 to the model +22 then we have all the steps of the eversion!

Fig. 16 First cuboctahedral eversion (Maubeuge 2000). On the picture: Philippe Charbonneau.

A first description of this eversion with annotated pictures is available in my article “Versions polyédriques du retournement de la sphère”[9], Retournement du cuboctaèdre[10] I wrote for the revue *L’Ouvert* [8] of the IREM of Strasbourg. In its “Retournement du cuboctaèdre” [5] François Apéry describes an other eversion which simplifies the previous one with the help of *linear interpolations*.

6 Conclusion

Three models with increasing complexity mark out the way towards the central stage of the eversion of the cuboctahedron. The halfway model represents an ideal point to start the study of the eversion. By building some models, the reader gives himself means to understand better what occurs during a sphere eversion. Animations with JavaView were also realized. Three of them were presented at the conference. This article reminds the long way of maturation and perseverance which preceded their achievement. It is also an encouragement to all those who think that they don’t understand maths to believe in their own capacities and to develop them!

Acknowledgments

I would like to thank all the persons who permitted the realization of this work. At first Bernard Morin for all his generous explanations, François Apéry, Claude-Paul Bruter, Jean Constant for their advices, Konrad Polthier and Ulrich Reitebuch for their help to use JavaView and ESMA for offering a platform from which to share with my peers, colleagues and the public at large.

References

[1] U. Brehm, *Minimal Polyhedral Versions of the Boy Surface,* Preprint Reihe Mathematik, Technische Universität Berlin (1988), no.189.

[2] B. Morin et J.-P. Petit, “Le retournement de la sphère”, *Pour la Science*, 15, (1979).

[3] F. Apéry, “Constructing Wire Models, Mathematics and Art, Mathematical Visualization” in *Art and Education* (C.P. Bruter ed.), Springer-Verlag, 2002, 179-200.

[4] F. Apéry, *Models of the Real Projective Plane: computer graphics of Steiner and Boy surfaces*, Wiesbaden: Vieweg, (1987).

[9] http://mathinfo.unistra.fr/fileadmin/upload/IREM/Publications/L_Ouvert/n094/o_94_32-45.pdf

[10] http://mathinfo.unistra.fr/fileadmin/upload/IREM/Publications/L_Ouvert/n095/o_95_15-36.pdf

[5] F. Apéry, *Le retournement du cuboctaèdre*, preprint, Institut de Recherche Mathématique avancée, U. Louis Pasteur, Strasbourg (1994).

[6] R. Denner, "Versions polyédriques minimales de la surface de Boy", *Quadrature* n° 70, Les Ulis: EDP Sciences (2008), 31-35.

[7] R. Denner, "Modèles centraux du retournement du cuboctaèdre ; généralisation", *Quadrature* n° 73, Les Ulis: EDP Sciences (2009), 32-39.

[8] R. Denner, "Versions polyédriques du retournement de la sphère, Retournement du cuboctaèdre", *L'Ouvert*, n^os 94, 95, IREM de Strasbourg (1999).

Retired mathematics teacher of the secondary school
Webmaster of the ESMA since 2011
ESMA, c/o Institut Henri Poincaré
11 rue P. & M. Curie
75321 Paris Cedex 05
rj.denner@evc.net

COLOSSAL CARDBOARD CONSTRUCTIONS

George Hart

Abstract

Impressive, large geometric sculptures can be made very cost effectively from cardboard by a group of people working together to cut the components and assemble them. In the process, the participants informally learn ideas about geometry and symmetry while developing team problem-solving skills and seeing first-hand how mathematics can be applied to art and design. Several examples are presented of "sculpture barn raisings" of this type that I have designed and led.

1 Introduction

I enjoy creating geometric sculpture both to challenge myself as an artist and because I feel mathematical art can have a pedagogical value for the viewer [1]. Carefully observing an unusual structure can put the observer into a mathematical mode of thinking - asking questions about the patterns and relationships inherent in the artwork. I have found that having a group of people help me assemble a geometric sculpture has an even stronger effect. Since the 1990's, I have been organizing events I call "sculpture barn raisings" in which I design a sculpture, fabricate the components from wood, metal, or other materials, and invite a community to participate in its construction [2][3][4][5][6]. Participants in these events get a hands-on introduction to the fun and creative sides of mathematics.

As awareness of my mathematical sculpture barn raisings has spread, more people have inquired about having me lead an event at their site. However, it can be difficult to find sufficient funding for the cost of purchasing and shaping permanent materials such as wood or metal. So in the past year, I have been experimenting by developing a series of designs which are suitable for fabrication in cardboard, which is much less expensive than wood or metal. Although it is not as strong, I can design for its properties and it has turned out to be sufficiently sturdy. Some of these sculptures have lasted many months and I expect they can continue to last much longer if treated gently.

2 Example Constructions

The examples presented here are all roughly spherical forms with icosahedral symmetry, in which all the parts are identical. That is not in any way essential to the larger ideas of this paper; it is simply my personal style in designing these works. Others might make large cardboard constructions based on very different mathematical ideas. My first experiment in this direction was at the Bridges Towson conference in July, 2012. For this, I designed a 1-meter diameter structure made of thirty painted rectangular components, each folded on a diagonal and joined using slots. See Figure 1 and the video of its assembly [7].

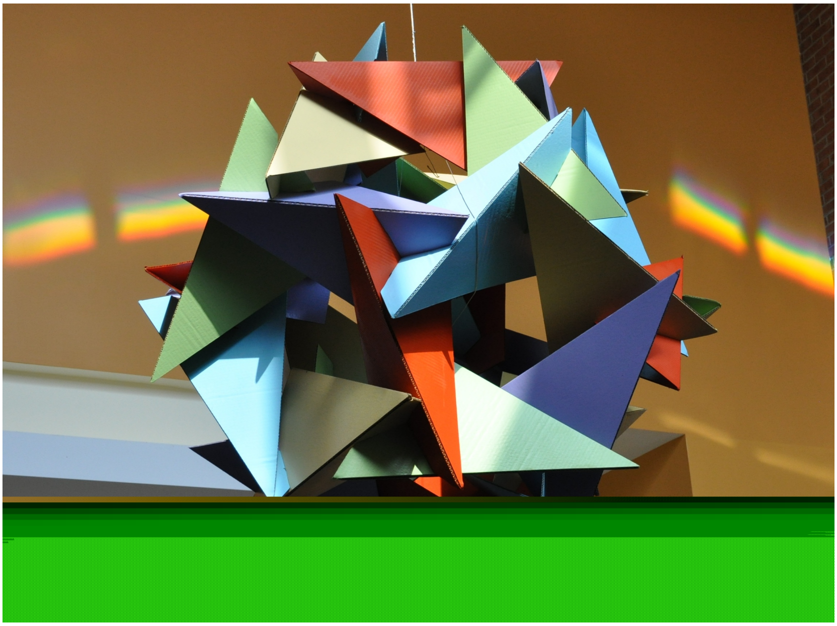


Fig. 1 One-meter cardboard construction made at the Bridges Conference 2012

Although the parts are simply slotted rectangles, the coloring aspect adds a certain richness and it was so successful that I went on to plan larger, more complex designs. Figure 2 shows a two-meter diameter cardboard construction built at a workshop I led at Southern Connecticut State University in October, 2012. A video of the assembly is online [8].

If there are several hours for the workshop, the participants can cut out the cardboard parts themselves as shown in [8]. If there is only an hour or so, then the parts need to be cut ahead of time. For cutting many identical parts, it is efficient to make stacks of the cardboard sheets, join them with “sheet rock screws” or clamps, and cut an entire pile at once with a saw. The part template

Fig. 2 Two-meter cardboard construction assembled from sixty identical components

is first traced on the top sheet of the stack. A band saw is ideal for this. With student groups, I have used a table-top scroll saw, which is not as fast as a band saw but is easy to transport to the venue and very safe compared to other types of saws. A spiral blade makes it easy to cut in any direction, which simplifies the process for people who have little shop experience. Younger participants can not use a saw, so their parts could be cut by others, perhaps ahead of time.

During the construction events, I can give detailed step-by-step instructions on how to assemble the parts. But if there is time, I can let the participants try to puzzle it out first. I have done this with teachers, leaving them confused for the start of the session. I think this is good to help them sympathize with what it is like to not understand the next step, as some of their students may sometimes feel. I also hope everyone then enjoys the “Aha!” experience when they do begin to understand the structure.

At some point in the workshop, I take time to explain the symmetry aspects of the design. It is useful to see the 5-fold, 3-fold, and 2-fold symmetry axes and use them as landmarks for adding additional parts to a partially complete construction. The chirality issue is also important to discuss. If the parts could be assembled in either left-handed or right-handed form, it is essential that everyone in the group make the same choice. A fundamental skill in much of mathematics is to learn to see patterns and extend them. In these designs, some visualization is required when geometric patterns are sometimes rotated or upside down from the exemplar. All of the designs shown here can be assembled in a modular manner at first. Groups can work in parallel assembling sub-units that combine into a larger structure. But due to the complexity of pieces getting in the way of each other, the final steps often require that individual pieces be inserted one at a time.

When designing for cardboard, there are many material issues to consider. Cardboard is available at low cost, but has relatively low strength, so is not suitable for long thin components. Corrugated cardboard has a grain, like wood, so folds easily in one direction while being more resistant to folding in the orthogonal direction. Aligning the template with the grain in the best direction can add significant strength to a part. Cutting with knives is not recommended as it is too easy to slip and cause injury. A band saw or scroll saw is much safer because the sharp part remains in one place. Using cardboard which is white on one side and brown on the other adds some visual interest while only slightly increasing the cost. In the US, a variety of sizes and thicknesses of cardboard sheets can be ordered and delivered through uline.com. Cardboard parts can be connected together with slots, clips, tie wraps, tape, and/or glue. Figure 3 shows a construction made from corrugated plastic election signs, which are freely available on the day after election day. This material is like cardboard in many ways but is slightly tougher and I used both slots and cable ties for the connections. This construction took place at Albion College, Michigan, in 2012. A video of its assembly shows the details, including the technique for using sheet-rock screws to hold together the stacks for cutting [9].

Fig. 3 Corrugated plastic construction made from recycled election signs at Albion College

Before making anything on a large scale, it it usually valuable to make a maquette. For most of these cardboard designs, I first made a paper scale model roughly 25 cm in diameter. The process is very useful for gaining insight into the structure and working out an efficient assembly sequence. It no doubt leads to a more robust final cardboard construction. Figure 4 and the video [10] show the paper model of a design which I later made in cardboard, 1.5 meters in diameter, with a group of teachers from Math for America in New York City. The cardboard construction is shown in Figure 5 and a video [11].

Fig. 4 30 cm paper model made in preparation for the large cardboard version of Fig. 5

Fig. 5 Cardboard construction made at Math for America teacher's workshop

After that, I worked with students in a workshop at Aalto University in Helsinki, Finland to make a larger version of the design from plywood sheets, shown in Figure 6 [12]. This suggests how any of these designs might be scaled up from paper to cardboard to larger dimensions using wood or metal, with corresponding changes to the connection system. In the paper version the parts are glued together; in the cardboard version the parts join with slots; in the plywood version cable ties are used.

After creating the design above, I continued to tweak it slightly in further versions. I changed the set of planes slightly so that the three pieces which meet at the exterior tips are orthogonal and surround an empty, small spherical space, as seen in the rendering of Figure 7.

Fig. 6 Plywood versions of Figs. 4 and 5 (left and right handed) at Aalto University

Fig. 7 Rendering before construction of design with orthogonal planes

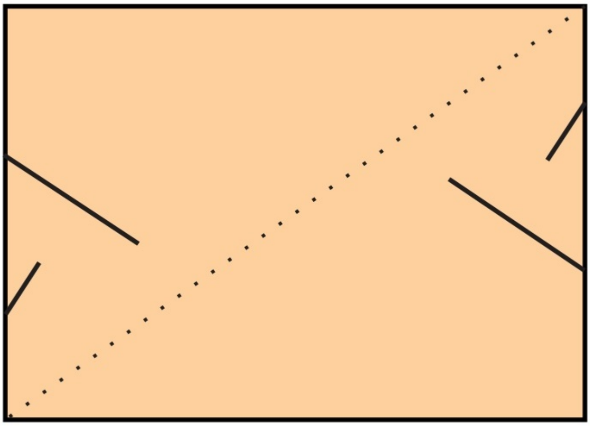
This is typical of how I see a computer model on the screen when designing and evaluating an idea, before committing to constructing it physically. I designed the examples shown here with the aid of a sculpture CAD program described elsewhere [13]. It understands how to arrange planes symmetrically in space and works with various symmetry groups. Working with it, I take into account material properties of cardboard and the connection system I envision.

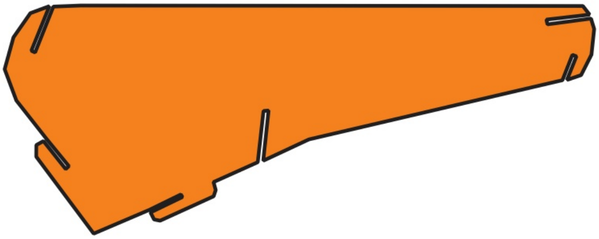
The two-meter cardboard construction of Figure 7 was built at a workshop I led at the Canada/USA MathCamp at Colby College, Waterville, Maine in July, 2013, shown in the video [14]. It is the first one where I tried using glue rather than slots. Brushing a thin layer of glue on folded flaps and holding them in place to dry with clamps takes longer than just sliding slots together, but it results in a stronger, permanent bond. So this is now my preferred method of joining cardboard. Participants must be cautioned not to use too much glue, or it takes much longer to dry. I then made one additional tweak for the version of Figure 8, built at St. Paul's School in Concord, NH, in November, 2013.

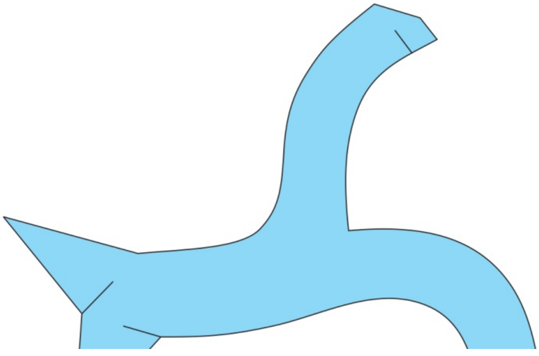
Fig. 8 Two-meter cardboard construction assembled from sixty identical components

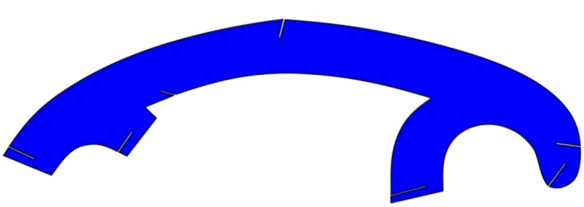
The change is not initially obvious in the completed structure as it only affects the interior. An inner opening which was triangular in Figure 7 was modified to be circular in Figure 8, in order to harmonize better with the circular exterior features. This change is more easily seen by comparing the last two templates of Figure 9.

Figure 9 provides templates for all the designs of this paper. Corresponding PDF files are available on my website [7]. To reproduce any of these constructions, I recommend printing the PDF at full scale and tracing it on to cardboard to make a master template which is then traced on to all the cardboard stacks to be cut. In some cases, marks must be made to indicate where parts are to join in the middle of a piece. I have found it is easy to mark these locations with small notches cut in to the edge of the template. When sawing, the notches are cut into the individual pieces and provide very clear alignment marks for guiding the assembly. Hopefully, the videos cited above give enough information for others to replicate the process.









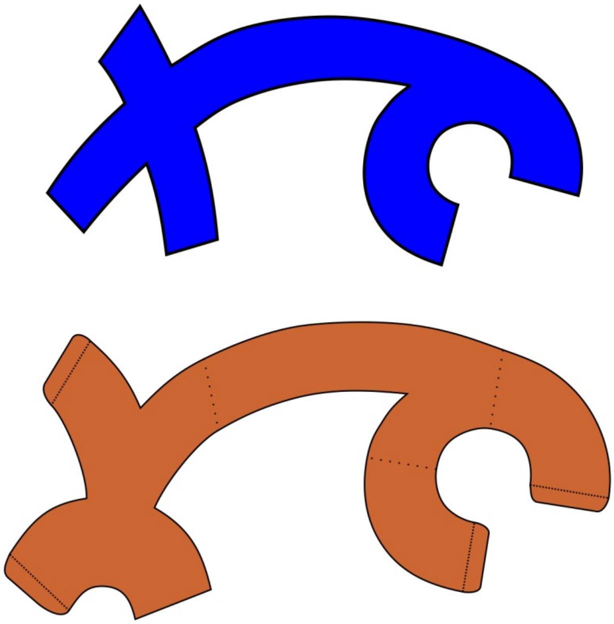


Fig. 9 Templates for the six designs above

3 Conclusions

I hope these constructions are not just visually engaging, but are replicated as fun challenging puzzles for developing collaborative problem-solving skills and a wider appreciation for the value of mathematics in art and design. While some artists are naturally possessive of their design ideas, viewing other's use of them as hurting their financial balance, I am happy to have others reproduce my designs if they want to make copies. Ugly corporate sculpture or shopping mall sculpture always depresses me when I see hundreds of people walk past without even a glance. So I find it personally gratifying when people not only look at my work but want to go to the effort of making their own copy. I always give permission freely and just ask that the copy be labeled "Design by George Hart" to distinguish it from the original instance where I was involved in the construction. It is great to see increasing interest and enthusiasm for mathematical art. I have additional designs on my drawing board and events scheduled, so I will be continuing to explore these colossal cardboard constructions in future work.

References

[1] G. Hart, http://georgehart.com

[2] G.Hart, "The Millennium Bookball," Proceedings of *Bridges 2000: Mathematical Connections in Art, Music and Science*, Southwestern College, Winfield, KS, July 28-30, 2000.

[3] G. Hart, "A Salamander Sculpture Barn Raising," Proceedings of *Bridges 2004: Mathematical Connections in Art, Music, and Science*, Southwestern College, Winfield, Kansas, July 2004.

[4] G. Hart, "Spaghetti Code: A Sculpture Barnraising," *Proceedings of Art+Math=X International Conference, University of Colorado*, Boulder, June 2005, pp. 88-92.

[5] G. Hart, "Barn Raisings of Four-Dimensional Polytope Projections," in *Proceedings of International Society of Art, Math, and Architecture 2007*, Texas A&M, May, 2007.

[6] G. Hart, "Comet!" in *Hyperseeing, Proceedings of ISAMA 09*, pp. 95-102, 2009.

[7] G. Hart, Cardboard Construction, http://youtu.be/\$_o$AYFPgR1OM

[8] G. Hart, Cardboard Conglobulation, http://youtu.be/wHOTeH94-2A

[9] G. Hart, Running for Office - Twenty Twel, http://youtu.be/DRQ-4f9e9KM

[10] G. Hart, Blue Paper Model, http://youtu.be/QhvfYucDeUA

[11] G. Hart, Teacher's Workshop, http://youtu.be/pI0AhEPcx0U

[12] G. Hart, Aalto, http://youtu.be/-mqu-QPH3dg

[13] G. Hart, "Symmetric Sculpture," *Journal of Mathematics and the Arts,* vol. 1, no. 1, pp. 21-28, March 2007.

[14] G. Hart, Giant Cardboard Construction.

George Hart
Stony Brook University
3 Stony Rd. Stony Brook
NY 11790 USA.
http://www.georgehart.com

RESILIENT CYCLIC KNOTS FOR STUDYING OF FORM-FINDING METHODS

Dmitri Kozlov

Abstract

The paper is dedicated to pedagogical uses of cyclic periodic knots made of resilient filaments. It describes one of my recent experimental workshops. The aim of the work was to design and build a large-scale model of transformable dome with tent covering and elaborate an algorithm of its form-finding process.

In the summer of 2012 I organized a workshop for the students of Moscow Architectural Institute. The aim of the workshop was to introduce to the students some of the new ideas and principles of physical form-finding based upon the properties of resilient cyclic knots. Today, when digital form-generation methods have become predominant in architectural and design education, the experimental exploring of alternative approaches to modeling and form-finding is especially important for students. The combination of physical and digital form-finding experiments helps them to understand the mathematical background common to both of these methods.

My form-finding method derives from the fact that cyclic periodic knots made of resilient filaments behave as kinetic form-finding structures [1]. Knots of this type must have a large number of physically contacting crossings functioning as the vertices of surfaces. The crossings slide along the resilient filaments and the filaments at the same time twist around their central axis. The waves on the filaments move and change their lengths to adapt to the current disposition of the contact crossings. Thanks to these properties the knots change their geometry as a whole and create vertex or point surfaces with an arbitrary Gaussian curvature. The complicated knots of this type I designated as NODUS-structures [2].

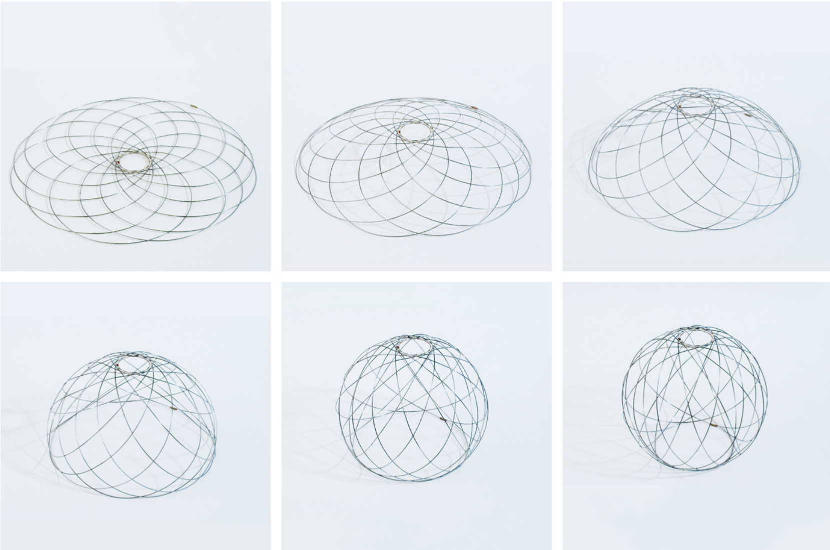


Fig. 1 Stages of transformation of resilient cyclic knot of steel wire

I took as a prototype one of my steel wire NODUS-structures that is a Turk's-Head-like non-alternating knot with 13 loops or bights and 12 leads (Fig. 1). Though this knot was made of a single piece of wire I proposed to the students to build the scale model of a large transformable dome of 5 meters in diameter with tent covering (Fig. 2).

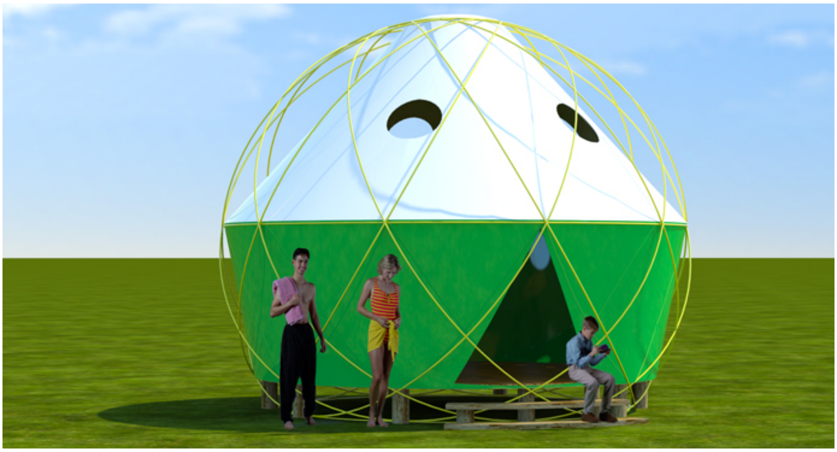


Fig. 2 Dome and tent design

It would be difficult and inappropriate to weave such a big structure with the single continual piece of filament material. Because we intended our model to copy not only the final shape of the dome but also the process of its assembly and erection (Fig. 3), we decided to divide it into 13 modules of equal length corresponding to 13 loops of the knot.

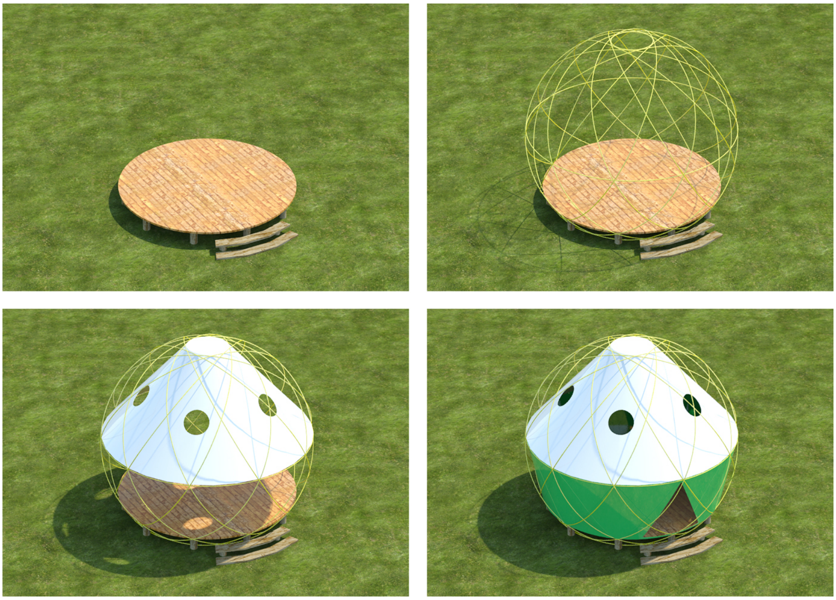


Fig. 3 Process of erection of the dome and tent

The material we used for the model was fiberglass wire around 4 mm in diameter coated in orange plastic divided into 13 modules of equal length corresponding to 13 loops of the knot. This lightweight and non-conductive material is very suitable for the modeling of resilient cyclic knots and links though its bending abilities are limited to some minimal radius. Values of radii less than this minimum may result in breakage.

We started our work by devising a detailed algorithm of the assembly process and depicting it as a series of pictures drawn on a computer. Each stage of the algorithm consisted of the order of connection of the corresponding module with the previous one and the order of its weaving through all of previous modules. The passing of the module in a crossing point over and under another module was marked as plus (+) and minus (-) signs correspondingly (Fig. 4). This sequence of over- and undercrossings was used as a reference guide to make the structure of the chosen non-alternating knot correctly.

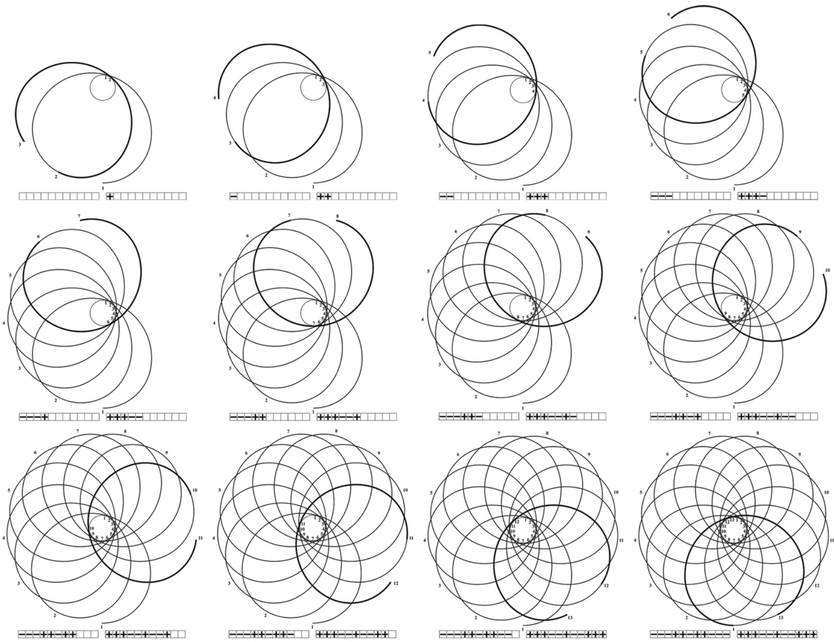


Fig. 4 Algorithm of assembly process

As a preliminary step we placed a ring in the center of the future structure and attached it to the floor to fix the central opening of the structure and tense it. Then we began the assembly of the structure, adding the modular elements of the knot, forming its loops and interweaving the modules according to the algorithm (Fig. 5).

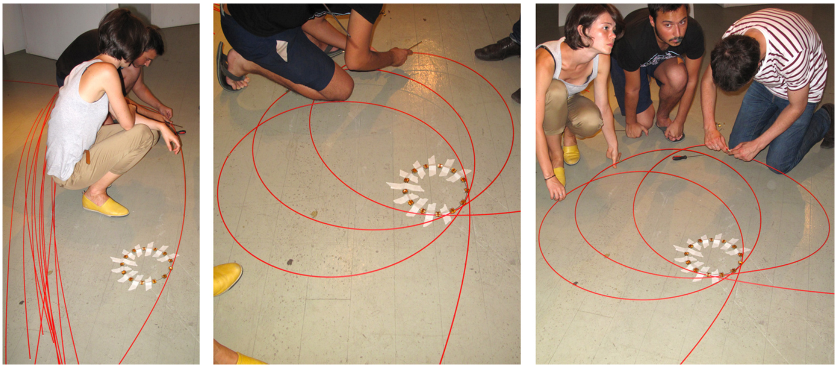


Fig. 5 Assembling of fiberglass wire modules into knotted structure

After we had finished all of the algorithm stages, we detached the central ring and tested the transformation of the structure (Fig. 6). It worked similar to the small wire model though it was not so stiff.

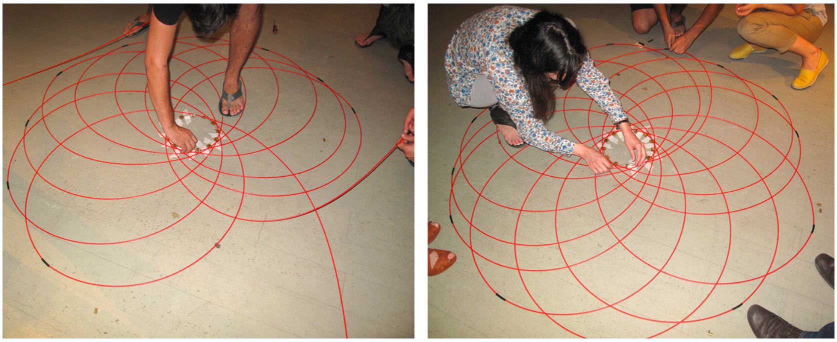


Fig. 6 Fixing central and peripheral openings of the structure with rings

Then we transformed our structure into the form of a truncated sphere and added another fixing ring on the peripheral opening (Fig. 7). As a result the whole structure became stretched inside the waves of the fiberglass wire and compressed at the contact crossings.

Fig. 7 Finding the final form of model

The next work was in finding the form of the tent covering and in searching of different ways how to fix the tent to the dome (Fig. 8).

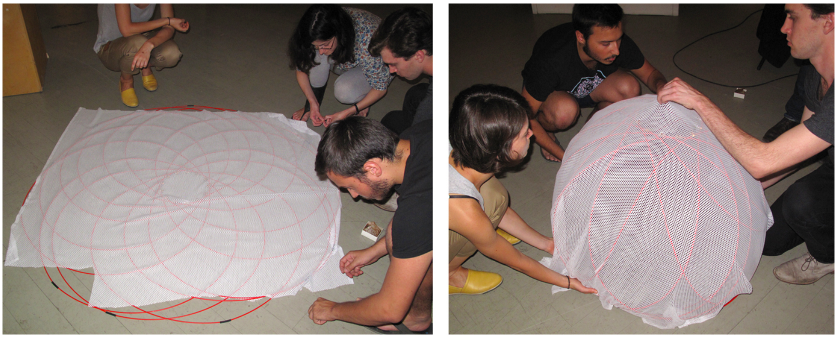


Fig. 8 Experiments with tent covering

Though for this experimental work was taken the simplest NODUS-structure that formed parts of spherical surfaces, it would be interesting to continue this work and try to design and build large scale structures of such forms as hyperboloids, tori, pretzels, self crossing, one-side and knotted surfaces, because the given method of form-finding may be extended to practically unlimited variety of surfaces [3]. This experiment may serve as good practice of physical modeling and form-finding for students as well as production of new pieces of kinetic art.

References

[1] D. Kozlov. "Knots and Links As Form-Generating Structures". In Bruter C., ed. *Mathematics and Modern Art: Proceedings of the First ESMA Conference*, Proceedings in Mathematics, Vol. 18, Berlin: Springer-Verlag, 2102, 105–115.

[2] D. Kozlov. "Polymorphous resilient flexible shaping structures "NODUS" for space and other extreme". In Bell, L., ed. *Proceedings of the First International Design for Extreme Environments Assembly* (IDEEA), Houston: IDEEA USA, 1991, 259–260.

[3] D. Kozlov. "Resilient Knots and Links as Form-Finding Structures". *Proceedings of Bridges: Mathematics, Music, Art, Architecture, Culture*, 2011, 179–186.

Dimitri Kozlov
Research Institute of Theory and History of Architecture and Town-planning
Russian Academy of the Architecture and Building Sciences
21-a 7th Parkovaya St.
Moscow
105264 Russia
kozlov.dmitri@gmail.com

ANGEL AND DEVILS ON TRIPLY PERIODIC POLYHEDRA

Douglas Dunham

Abstract

M.C. Escher created a version of his “Angels and Devils” pattern in each of the three classical geometries. In this paper we extend this idea to patterns on triply periodic polyhedra, thus filling combinatorial gaps in Escher’s work.

1 Introduction

In this paper we show new “Angels and Devils” patterns on triply periodic polyhedra that were inspired by related patterns of the Dutch artist M.C. Escher. Triply periodic polyhedra have translation symmetries in three independent directions in Euclidean 3-space. Figures 1 and 2 show finite pieces of two such polyhedra decorated with angels and devils. Each of the polyhedra we discuss is composed of copies of a regular polygon, with more of them around each vertex than would be possible in the Euclidean plane, so we consider them to be hyperbolic. These polyhedra thus have negative curvature, and are related to regular tessellations of the hyperbolic plane. Similarly, the patterns we place on these polyhedra are related to patterns of the hyperbolic plane that are based on the corresponding tessellations. We first review regular hyperbolic tessellations and triply periodic polyhedra, and the relation between them, which extends to patterns on the respective surfaces. Then we analyze Angels and Devils patterns on two polyhedra.

2 Regular Tessellations and Triply Periodic Polyhedra

We use the Schläfli symbol $\{p,q\}$ to denote the regular tessellation formed by regular p-sided polygons or p-gons with q of them meeting at each vertex. If $(p-2)(q-2) > 4, \{p,q\}$ is a tessellation of the hyperbolic plane (otherwise it is Euclidean or spherical). Figure 3 shows the tessellation $\{4, 5\}$ superimposed on a pattern of angels and devils in the Poincaré disk model of hyperbolic geometry. We will be interested in infinite, connected semiregular triply periodic polyhedra.

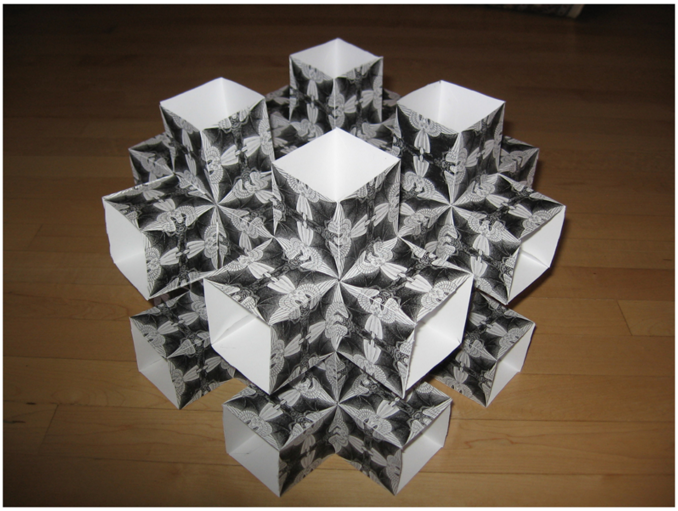


Fig. 1 Angels and devils on a piece of the $\{4,6\}$ polyhedron.

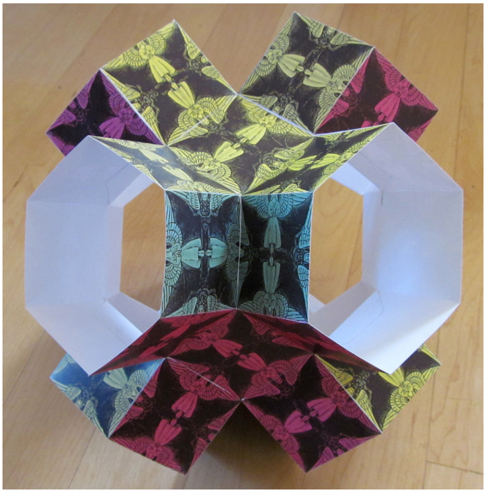


Fig. 2 Angels and devils on a piece of a $\{4,5\}$ polyhedron.

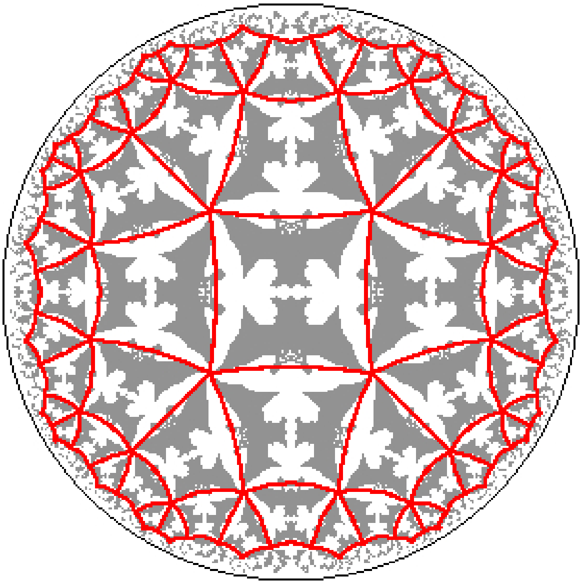


Fig. 3 The $\{4,5\}$ tessellation superimposed on a pattern of angels and devils.

Such a polyhedron has a p-gon for each of its faces, q p-gons around each vertex, translation symmetries in three independent directions, and symmetry group that is transitive on vertices—i.e. it is uniform. We extend the Schläfli symbol $\{p,q\}$ to include these polyhedra (however different polyhdera can have the same $\{p,q\}$). Figures 1 and 2 show pieces of {4, 6} and {4, 5} polyhedra. There is often a surface that is intermediate between triply periodic polyhedra $\{p,q\}$ and the corresponding regular tessellations $\{p,q\}$. First, these periodic polyhedra are approximations to triply periodic minimal surfaces (TPMS). Figure 4 shows a piece of Schoen's I-WP TPMS that corresponds to the {4, 5} polyhedron of Figure 2 [2]. Second, each smooth surface has a universal covering surface: a simply connected surface (the sphere, Euclidean plane, or hyperbolic plane) with a covering map onto the original surface. Since each TPMS has negative curvature (except for possible isolated points), its universal covering surface does too, and thus has the same large-scale geometry as the hyperbolic plane. In the same vein, we might call a hyperbolic pattern based on the tessellation $\{p,q\}$ the "universal covering pattern" for the related pattern on the polyhedron $\{p,q\}$. The pattern of Figure 3 is the universal covering pattern for Figures 2 and 6.

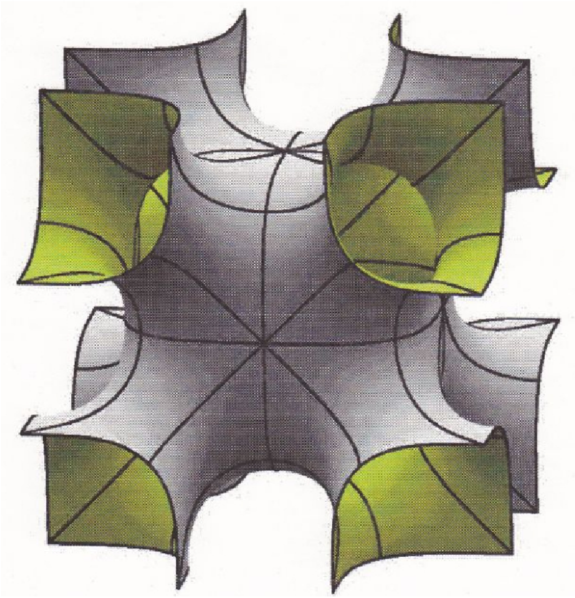


Fig. 4 A piece of Schoen's I-WP TPMS which corresponds to the $\{4,5\}$ polyhedron.

3 Angels and Devils on the {4,5} and {4,6} Polyhedra

Figure 1 shows the {4, 6} polyhedron, the simplest triply periodic polyhedron, which is based on the tessellation of 3-space by cubes. The solid within the {4, 6} consists of invisible "hub" cubes that are connected by (visible) "strut" cubes, each hub having a strut on each face, and each strut connecting two hubs. The Schwarz P-surface is the corresponding TPMS – it is basically a smoothed out version of the {4, 6} polyhedron [2].

Figure 2 shows a piece of a {4, 5} polyhedron, which can also be described by the solid within it. That solid consists of truncated octahedral hubs (the square faces of which are visible) with their hexagonal faces connected by regular hexahedral prisms as struts. Figure 2 shows one hub and its 8 connecting struts. As mentioned above, Schoen's I-WP surface is the corresponding TPMS. Figure 5 shows a {4, 5} polyhedron that is actually the same polyhedral surface as that of Figure 2 [1]; Figure 2 shows the outside and Figure 5 shows its "complement", the inside. The Figure 5 polyhedron is made up of cross-shaped units (shown in different colors), each of which is a cube with four equilateral triangular prisms on it. Figure 6 shows an angels and devils pattern on that polyhedron.

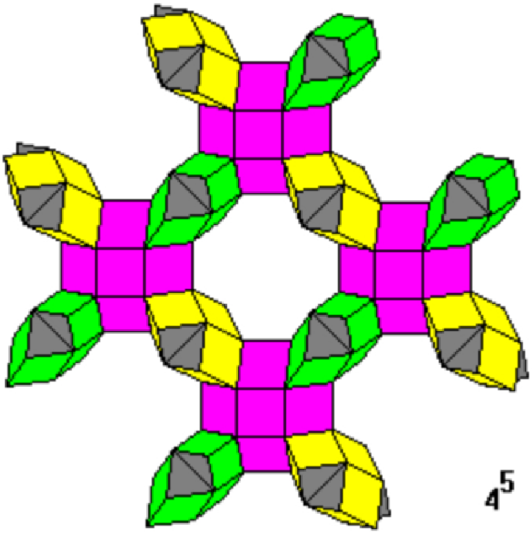


Fig. 5 A piece of the $\{4,5\}$ polyhedron that is the "complement" polyhedron to that of Figure 2.

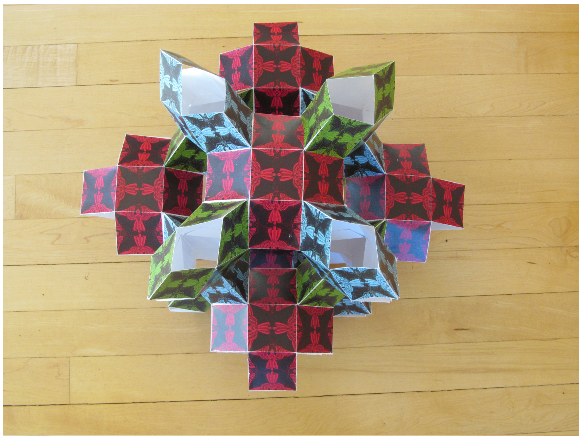


Fig. 6 A piece of the $\{4,5\}$ polyhedron that is the "complement" polyhedron to that of Figure 2.

Escher's "Angels and Devils" pattern, the only one he realized in each of the three classical geometries, were based on the {4, 3} (spherical), {4, 4} (Euclidean plane), and {6, 4} (hyperbolic) tessellations. The patterned {4, 5} polyhedra of Figures 1 and 6 thus fill a "gap" between Escher's {4, 4} pattern and the patterned {4, 6} polyhedron of Figure 1.

References

[1] S. Dutch, Wells, "*Hyperbolic Tessellations*", http://www.uwgb.edu/dutchs/symmetry/hypwells.htm

[2] A. H. Schoen, http://schoengeometry.com/e-tpms.html

Douglas Dunham
Department of Computer Science
University of Minnesota
Duluth
MN 55812-3036 USA
ddunham@d.umn.edu

NONLINEAR MUSICAL ANALYSIS AND COMPOSITION

Renato Colucci, Gerardo R. Chacón,
Juan Sebastian Leguizamon C.

Abstract

We discuss the application of Nonlinear time series analysis in the context of music analysis. We comment the results presented in [4] and give some ideas for further investigation (see [5]). Moreover we show how these techniques can be used to produced original music with both artistic and pedagogical purposes.

1 Introduction

The methods of non linear time series analysis has been widely used in studying many natural and social phenomena (see for example [7], [11] and [9]). The most important tool is Takens' theorem (see [12]) that lets us reconstruct the whole phase space by considering the data in a proper m-dimensional Euclidian space. If we denote by $\{x_i\}_{i=0}^{N}$ the original data set then the m dimensional vectors, called m-histories, are constructed in the following way:

$$\begin{aligned} &h_1 = (x_0, \dots, x_{m-1}) \\ &\dots \\ &h_{N-m+1} = (x_{N-m+1}, \dots, x_N) \end{aligned}$$

The dynamics on the pseudo-phase space $\mathcal{H} = \{h\}_{i=1}^{N-m+1}$ is diffeomorphic to the dynamics on the true attractor of the system. Then analyzing the data on the embedding space is it possible to obtain the estimation of many important quantities such as the dimension of correlation or Liapunov exponents. In [8] and [4] a way to apply these techniques has been proposed in the context of musical analysis (see section 2). In the present paper we continue the implementation of nonlinear techniques in the context of music. In particular we give some examples of application of prediction algorithm to simulate a musical styles (see section 3). In section 4 we show that prediction algorithm can also be used to produce original music. In particular we suggest a method of random interpolation of a set of m-histories with another set corresponding to two different composition (here we chose Prelude of Suite I for Cello by J.S. Bach and Sequenza IXb for Alto Saxophone by L. Berio). The new set of m-histories will be used to make prediction and the result will be some mixture of the two compositional styles. There are many possibilities to apply this methods to compose original music both for artistic and pedagogical purposes. In the last section we give some remarks and suggestions for future investigations.

2 Discussion of previous results

In [4] the authors analyzed three different compositions and discussed the technical difficulties while applying these techniques in the context of musical analysis. The compositions analyzed, Prelude of Suite n. 1 for cello solo (1720-1721) by J.S. Bach (1685-1750) (see [13]), Syrinx (1913) by C. Debussy (1862-1918) (see [6]) and Tenor Saxophone Solo from Acknowledgement (from the Album A Love Supreme, 1964) by J. Coltrane (1926-1967) (see [3]) are all compositions written for an instrument. The authors of the paper made an identification of a solo musical composition (see [8]) with a time series and apply some time series techniques in order to analyze them. As a summary of the results of [4], the three different compositions written using different styles such Baroque Counterpoint (Bach), Free Modern Composition (Debussy) and Modal Jazz (Coltrane), satisfy the following inequalities concerning embedding dimension (m) and correlation dimension (D):

$$m_{\text{Debussy}} \leq m_{\text{Bach}} < m_{\text{Coltrane}}, \tag{8.1}$$

$$D_{\text{Debussy}} < D_{\text{Bach}} < D_{\text{Coltrane}}. \tag{8.2}$$

This suggest that the music of John Coltrane can be described by using more patterns/variable with respect to music of Debussy but looking at the inequality regarding Liapunov exponents (L):

$$L_{\text{Bach}} < L_{\text{Coltrane}} < L_{\text{Debussy}}, \tag{8.3}$$

we guess that the patterns of Debussy are arranged in a more unpredictable way with respect to that of Coltrane's.
Then it is natural to ask if the above reasonable results are due only to the great difference between the three compositions or if these methods really work in general cases. It could be interesting to ask if it is possible to catalogue music by these nonlinear techniques depending on styles, genres, composers, etc.

Here we present, as a first approach, the analysis of the whole Suite No. 1 in G major, BWV 1007 from "Six suites for Cello" by Johann Sebastian (1717-1723). The structure of the movements of the suite are the following:

- Prelude;
- Allemande;
- Courante;
- Sarabande;
- Minuet;
- Gigue.

In table 2 we represent the values of the standard deviation (σ), of the embedding dimension (m), correlation dimension (D) and Liapunov Exponents (L) of the six movements of the first suite.

Movements	σ	**m**	**D**	**L**
1	5.62476483	7	2,405810844	0,245919883
2	5,59381701		2,626908577	0,194686888
3	5,33603512		2,276816981	0,12682068
4	4,47777764		1,635210943	0,059480429
5	5,531770331		2,045760793	0,160438678
6	4,662153419		2,429508864	0,125400802
Mean Value	5,2043863920		2,2366695	0,15212456

We observe that with the exception of the fourth movement, the other movements share similar values of the analyzed quantities. A deeper analysis on the problem of cataloguing music can be find in [5] where the authors use different techniques such as recurrence analysis and pattern analysis.

3 Prediction algorithm

The most important goal of the nonlinear time series analysis is to make predictions in order to understand the future behavior of a complex system. In the context of music analysis it is interesting to ask if it has sense to consider a prediction algorithm for a musical composition. In our opinion those algorithms could be used to simulate a musical style or to produce new music.
We give an example of that for the music of Coltrane, a complete study of problem of prediction can be found in [5].
We consider the following algorithm (see [1])

$$y^m_{t+1} = \sum_{i=1}^{N} \left[\hat{y}^m_{k+1} - \hat{y}^m_k + y^m_t \right] \omega_k(y^m_t, \hat{y}^m_k), \tag{8.4}$$

where y^m_t is the last $m-$history of our data set, y^m_{t+1} is the $m-$history that we want to predict, the points

$$\hat{y}^m_k \in B_\varepsilon(y^m_t), \quad k = 1, \ldots, N, \tag{8.5}$$

are the neighbors of y_t^m contained in the neighborhood B_ε and $\hat{y}_{k+1}^m$ are the next points of $\hat{y}_k^m$. The weight functions are given by the following expression

$$\omega_k(a,b) = \frac{K_h(\|a-b\|)}{\sum_{k=1}^{N} K_h(\|a-b\|)}, \tag{8.6}$$

with the Gaussian Kernel:

$$K_h(x) = \frac{1}{h} K\left(\frac{x}{h}\right), \qquad K(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}. \tag{8.7}$$

Once a set of $m-$histories has been produced it is possible to construct the one dimensional prediction by considering only the first element of each $m-$history. In figure 1 we represent a portion of the Coltrane's solo from bar 97 to bar 104 while in figure 2 we present a prediction with $\varepsilon = 30$ and $h = \frac{1}{2}$.

Fig. 1 Coltrane's Solo

Fig. 2 Prediction with $\varepsilon = 30$ and $h = \frac{1}{2}$

We consider the algorithm of prediction only for the tones and put almost all the original values of the tones (for prediction for both tones and values see [5]). We observe that the most used scale by Coltrane is the Pentatonic of G minor while in the prediction all the predicted tones belong to the Pentatonic scale of C minor which is very close to that of G minor, moreover Coltrane uses this pentatonic scale during the continuation of his solo. These simple examples show that prediction algorithm also work in the context of music.

4 Original Music Production

In this section we give an example of how to use these prediction algorithm to create new music. The possibilities of the algorithms have only the limits of author's creativity. We propose a method based on the interpolation of one composition with another. We consider the Sequenza IXb by L. Berio for Alto Saxophone and again The prelude of the first Cello Suite by J.S. Bach. We cut the Sequenza at part D (included) since this is an homogeneous part (see [10]) and the embedding dimension results to be 5. Then we consider the 5-histories of the Prelude, we note that the embedding dimension is 7. We interpolate randomly the 5-histories of the Sequenza with the 5-history of the prelude in the following way: we consider the first 5-histories of Sequenza and the algorithm randomly decides to insert the first 5-history of Bach or to continues with the second 5-history of Sequenza. We note that this procedure mixes the 5-history without changing the order in which the histories of Bach and Berio appear. When the new set of 5-history is constructed we are ready to make a prediction. Since almost all the tones of the Prelude have the same value of 16th note, for simplicity in this case we predict only the pitch and put all 16th note as in the prelude. A simple way to introduce variations in the rhythm, without using prediction algorithms for the duration of the tones, is to randomly assign values to the pitches. This method is more suitable for contemporary music, while for classical music it would be necessary to use more restrictions. It is possible to use also irregular groups in the style of Sequenza but the algorithm that assigns values should need some constraints. More examples and discussion on prediction of the values can be find in [5].
In figure 3 below we represent the result of the random interpolation (in a five dimensional embedding space) of the Sequenza with the Prelude using $h = 0.5$ and $\varepsilon = 30$. It is interesting to note that if we want to change the roles of the two compositions we have to change the embedding space. We randomly interpolate the 7-histories of the Prelude with 7-histories of the Sequenza, the result is presented in figure 4 below. Again we use $h = 0.5$ and $\varepsilon = 30$.

The production of new music will require more investigations and the experimentation of musicians. We remark that it is possible to interpolate a composition with original (written) or random material, or in the contrary, we could start from some original or random material and interpolate. This method would work also for live performances in which, due to the random interpolation, a different musical sheet could be produced at each concert.

Fig. 3 Random interpolation of the Sequenza IXb using the Prelude of Suite I

5 Conclusion

In the present work we have discussed the results of [4] and gave some previews of the investigations about cataloguing and simulating musical styles which are the main topics analyzed in the forthcoming paper [5]. Moreover we give some examples of how these methods can be used in a variety of ways to produce original music. Another technique from the nonlinear time series analysis that we consider could be useful for cataloguing and producing original music

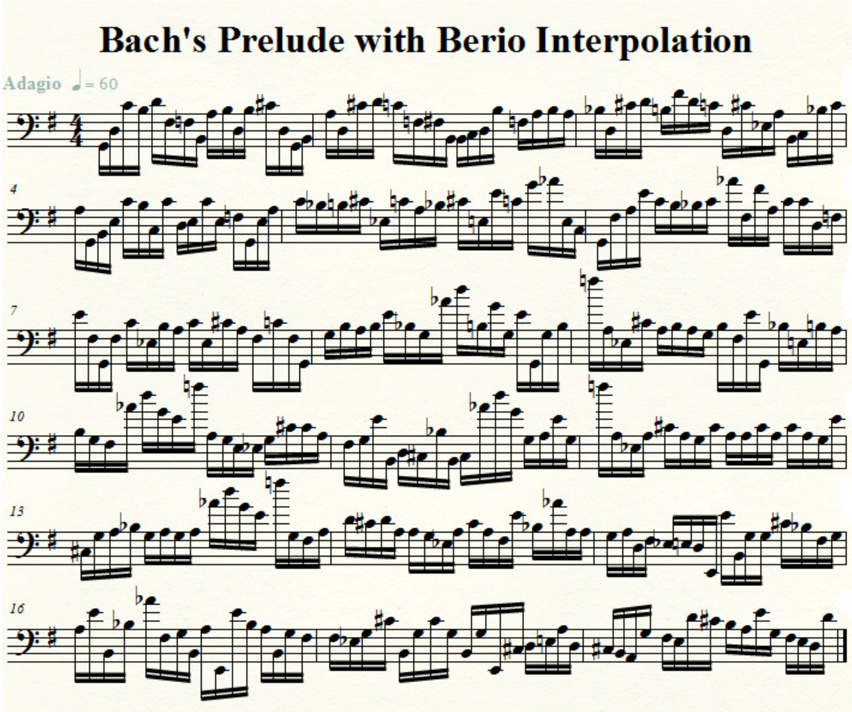


Fig. 4 Random interpolation of the Prelude of Suite I using Sequenza IXb

is given by the pattern recognition algorithms. In particular, we consider the machine learning method to recognize patterns (see for example [2]) should be explored in this setting and combined with the other techniques.

References

[1] A. Basharat and M. Shah, *Time series prediction by chaotic modeling of nonlinear dynamical systems*. Computer Vision, 2009 IEEE 12th International Conference, 2009.

[2] C. Bishop. *Music: Pattern Recognition and Machine Learning*. Springer Verlag, 2006

[3] J. Coltrane. *A Love Supreme: Tenor Saxophone*. Hal Leonard, 2003.

[4] Renato Colucci, Gerardo R. Chacón and Sebastian Leguizamon C. *Some Ideas on Nonlinear Musical Analysis*. Applied Mathematical Sciences, Vol. 7, 2013, no. 26, 1283-1301.

[5] Renato Colucci, Gerardo R. Chacón and Sebastian Leguizamon C. *Cataloguing and simulating music by nonlinear techniques*. Preprint.

[6] C. Debussy. *Syrinx, pour Flûte seule*. J. Jobert, 1927.

[7] A. Facchini, C. Mocenni, N. Maewan, A. Vicino and E. Tiezzi. *Nonlinear time series analysis of dissolved oxygen in the Orbetello Lagoon (Italy)*. Ecological modelling 203, 2007.

[8] C. Madden. *Fractals in Music*. High Art Press, Salt Lake City 2007 2nd edition.

[9] R. Perli and M. Sandri. *La ricerca di dinamiche caotiche nelle serie storiche economiche: una rassegna*. Note Economiche del Moten dei Paschi di Siena, Anno XXIV n. 2 (1994).

[10] F. Sbacco. *Appunti del corso di ansalisi musicale*. Unpublished, 2008.

[11] S. Sun and X. Xing. *The Research of the Fractal Nature between Costs and Efficacy in the Brain Vascular Disease*. Journal of Applied Mathematics Volume 2012, Article ID 171406, 12 pages doi:10.1155/2012/171406.

[12] F. Takens. *Detecting Strange Attractors in Turbulence*. Lecture Notes in Math. Springer, New York, 1981.

[13] A. Winold. *Bach's Cello Suites: Analyses and Explorations (Vol. 1 & 2)* . Indiana University Press, 2007.

Renato Colucci
Lecturer (Assistant Professor)
Xi'an Jiatong-Liverpool University
111 Ren Ai Road
Dushu Lake Higher Education Town
SIP, Suzhou
215123 China
renatocolucci@hotmail.com

Gerardo R. Chacon
Assistant Professor
Gallaudet University
800 Florida Avenue NE
Washington, DC 200002
USA
grchacon@gmail.com

Juan S. Leguizamon
Pontificia Universidad Javeriana
Carrera 7 N.43-82
Bogota
Colombia
sebasgta@gmail.com

EXPERIMENTAL MATHEMATICS

Francesco De Comité

Abstract

Computer tools (ray-tracing software, 3D modeler), new technologies (laser cutting, 3D printing), new communities (fablabs), increasing computing power and improved graphic screens: all together make it possible for the mathematicians to give life to their ideas. Moreover, having described basic mathematical objects, one can play with their parameters and just see what arises. This paper illustrates this fact through several examples: anamorphoses, cardioid based structures and circle packings.

1 Introduction

We can assume, with strong confidence, that Gaston Julia (1893-1978) never saw the image of a Julia set. Felix Klein (1849-1925) and Robert Fricke (1861-1930) drew only one representation of a limit set [8]. Some thirty years ago, on the 1st of March 1980, Benoît Mandelbrot first saw a blury image of the fractal set christened after him appearing on his printer. He then spent several weeks making new images, to understand the shape he just discovered [14]. Nowadays, anyone can write or download a program or a smartphone application, running orders of magnitude faster, zooming to scales out of reach at Mandelbrot's time. Playing with mathematical objects is easier today, scientists can experiment more sophisticated structures in reasonable time. Any mathematician with elementary programming skills can imagine a new shape, a new curve, translate it into a program and see it appearing within seconds on his high resolution screen, using a standard computer. Exploring the field of mathematical objects becomes easy and affordable due to several factors:

- The increasing power of processors, which makes calculations easier to perform.
- High resolution of computer screens, which gives precise and realistic images.
- High level programming languages which let the programmer focus on the heart of the problem to solve.

- Specialized computer applications like ray-tracing software or 3D modelers, when mastered, can also alleviate the mathematician's work. Those softwares are available in open-source and free versions.
- Laser-cutting machines, 3D printers are valuable tools to help making real-world versions of mathematical shapes.

In the rest of the paper, I will present several categories of works achieved by using part or all of those tools, and I try to show how, from the initial object I wanted to visualize, I was able to tune the parameters in order to obtain a whole family of related objects.

2 Anamorphoses

2.1 Definition

Anamorphoses are distorted images (resp. objects) needing to be seen through a mirror, and from a specific point of view, in order to reconstruct the original image (resp. objects). Anamorphoses appeared in Europe at the time of Renaissance, when artists and scientists discovered the laws of perspective, and independently in China during the XVIth century. A complete history of anamorphosis can be found in Jurgis Baltrušaitis' *Anamorphoses ou Thaumaturgus opticus* [3].

There are different methods for producing anamorphoses, either analytical, empirical or hybrid.

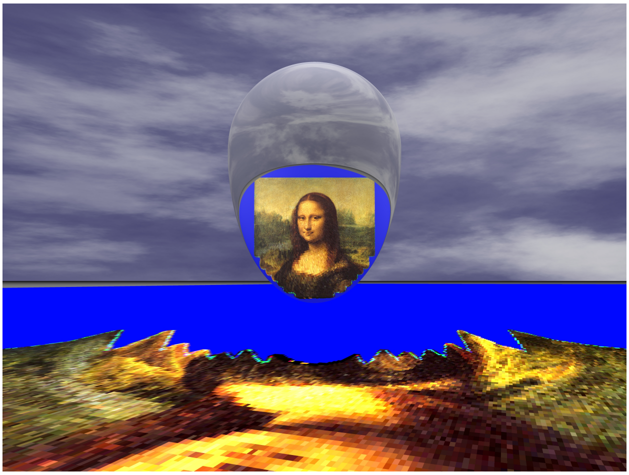


Fig. 1 Anamorphosis in an egg-shaped mirror

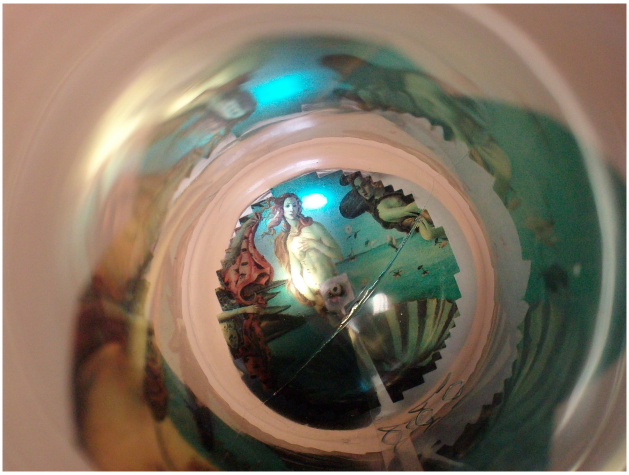


Fig. 2 Distorted image on a cylinder

2.2 A Computer Method and its Evolution

In [5], I described a procedure for testing and constructing catoptric anamorphoses in the general case. The setting-up of an anamorphosis needs three items: a mirror, an observer, and the locus where the distorted image lays, which I will name *surface of distortion*. This method can compute the distorted image corresponding to an anamorphosis setting and print it, in order to obtain a real size version of the anamorphosis. In [5], the surface of distortion was either a flat or a developable surface. Extensions of the method can also help define anamorphoses where the surface of distortion is no longer developable: we then lose the possibility of printing the distorted design, but are still able to achieve it by directly drawing on the surface, using the information returned by the program. In October 2010, James Hopkins, a British sculptor [9], asked me whether it would be possible to compute the shape of a three-dimensional wired form sculpture which would represent a chair when seen through a spherical mirror. This led to an improvement of the original method [6], where the surface of distortion is replaced with a set of volumes of distortion. The new method computes the distorted image of a line (either a line segment or a circle ark), and gathers all those images in a single three-dimensional anamorphic sculpture. Then exporting the results of those computation to a file, we can build a real-world three-dimensional object, using 3D printing techniques. Figure 3 shows an example of this process. Yet another improvement of the method is under development, in collaboration with James Hopkins, allowing the definition of 3D distortion of plain quadrilaterals (see figure 4). The related real-world sculpture is under construction (november 2013).

In conclusion, when a method, and the tools used in it are well mastered, improvements and evolutions are easy to implement. One can imagine possible improvements, then test them first virtually and validate them with real-world achievements.

Fig. 3 3D anamorphic sculpture

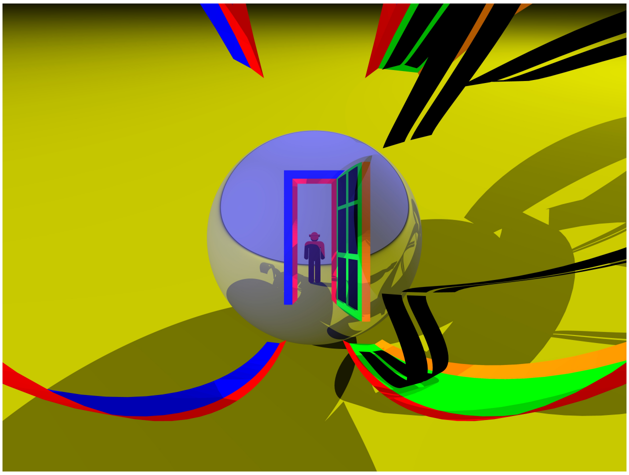


Fig. 4 Distorted quadrilaterals (preliminary try)

3 Cardioidal variations

The cardioid is a very old and known curve, which can be defined in several manners. Pedoe describes a method for constructing a cardioid as the envelop of a set of circles:

- Draw a circle and choose a point on its circumference.
- Draw circles with centers lying on the initial circle, and passing through the chosen point.
- The envelop of this set of circles is a cardioid [12] (see figure 5).

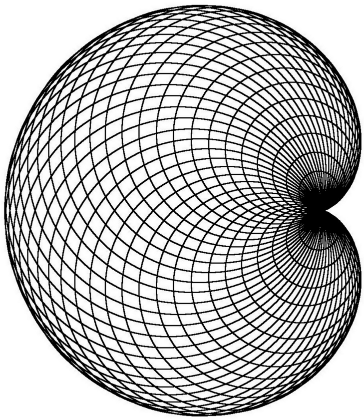


Fig. 5 Pedoe method for cardioid

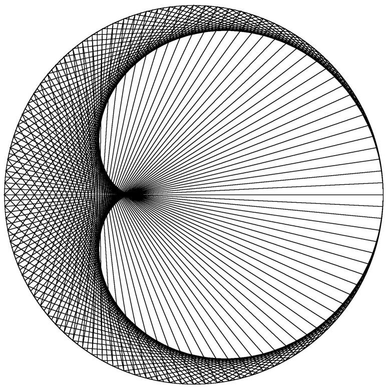


Fig. 6 String method for cardioid

But the result is flat. What if we rotate each circle in the third dimension, with an angle depending on its radius ? The function relying the rotation angle to the radius can be arbitrary, and each choice defines a different final shape (see figure 7). The shape is not difficult to code, once drawn. We can output the information (circle centers, radii, angles) and use that to obtain a real three-dimensional version of the virtual drawing. The experience of turning this 3D cardioid in your hands is still stronger than seeing it on the screen. Each angle of vision makes it look different, and the observer find new symmetries each time he moves it.

Playing with the function defining the rotation of each elementary circle, one can obtain an infinity of different shapes, all very different from each other. The best way to investigate this family of shape is either intensive tests, or animations. Intuition alone may miss interesting structures.

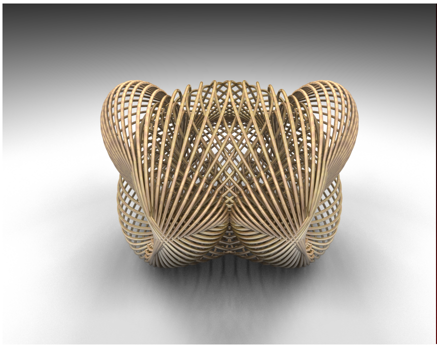


Fig. 7 Variation from Pedoe method

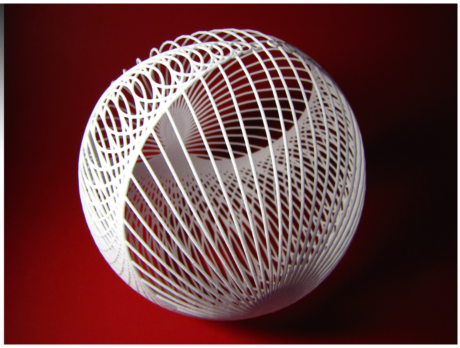


Fig. 8 Variation from string method

Another way to draw a cardioid is the following algorithm, whom result is shown on figure 6:

- Draw a circle.
- All around the circumference of this circle, draw n equidistant points.
- Numbering those points $1 \cdots n$, draw line between points i and $(2 \times i)$ mod(n)

Once again, the result is flat: we can now try to replace each line segment with a torus whose diameter equals the length of the segment, and that produces the object of figure 8. Surprisingly, the external shape of this object is a simple sphere. Simple mathematical reasoning might prove this, but the fact that it was *discovered* while drawing it is a strong argument in favour of experimental mathematics. A lot of explorations are still to be done: for example, what if the couple of points defining a line segment is changed from $(i, 2 \times i)$ to $(k_1 \times i, k_2 \times i)$? There is so much to explore that even those easy programming tasks are not yet achieved.

4 Circle packings

Circle packing can be seen as the art of placing tangent circles on the plane, leaving as little unoccupied space as possible. Circle packing has been (re)introduced by William Thurston [15] in 1985. Kenneth Stephenson developed its study in [13]. In this section, I will show how one can use different ways of producing circle packings, together with different geometric transformations that preserve the tangency property of the arrangement, in order to produce elegant and appealing images. For sake of aesthetical homogeneity, I will only consider

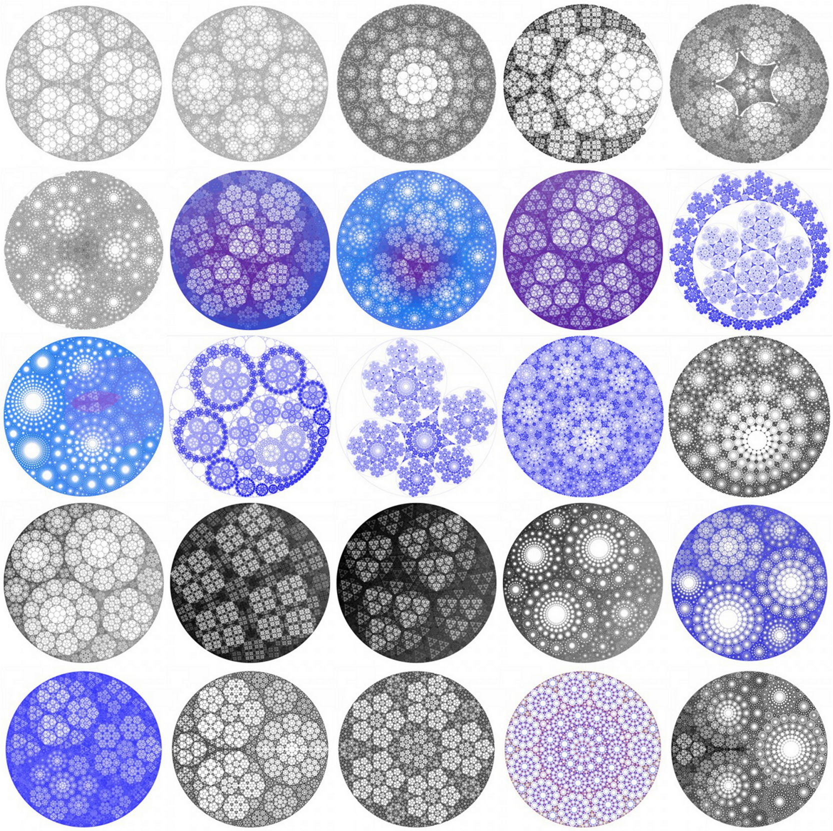


Fig. 9 Circle packing experiments

packings of tangent circles included in one external circle, also part of the tangency pattern. I mainly use two operations: defining Steiner chains, and using "Apollonian Gasketization" to fill the gaps between the generated circles. At Bridges 2012, Inglis and Kaplan [10] presented a method to produce fractal circular rings of tangent circles, where spaces are filled with Apollonian circles. Their algorithm shares some similarity with ours, but is in some ways more restrictive.

4.1 Steiner Chains

Steiner chains are chains of tangent circles, each of those circles also tangent to two fixed and non intersecting circles. Steiner chains are obtained by first constructing the easy solution: n circles forming a chain between two concentric circles, then using a circle inversion to distort the arrangement (since circle inversion preserves tangency). The way one chooses the inverting circle is a parameter of the design, which can be used to create an infinity of different shapes. The number n is also a parameter that leads to different solutions.

4.2 Apollonian Gasket

Apollonian gaskets are obtained by recursively filling the gaps between three tangent circles with a circle tangent to all those three circles. The standard Apollonian Gasket starts with three equal circles inside a fourth one. In our procedure, we consider filling the gaps between any set of three tangent circles.

4.3 Putting Things Together

The procedure used to generate circle packing patterns can be summarized in one sentence: *"Each time you generate a new circle, fill it with a Steiner chain, then fill the gaps by mean of gasketization"*. Each *steinerization* might have its own parameters. Such a simple construction rule produces a very rich set of different patterns. We can still add more diversity by applying at the end of the procedure a geometric transform that preserves the tangency property, like Möbius transform[1] or circle inversion. A complete exploration of the graphical potentialities of Möbius transforms can be found in Mumford's Indra's Pearls [11]. Figure 9 shows some variants. More are available in [7], and still more of them are to be discovered.

5 Conclusion

Using those three examples, I wanted to show that wide fields of mathematical art can be cleared by random exploration. The programmer can launch his program, looking at the image appearing on his screen. He can first compute a low resolution image, or abort the drawing if he is not satisfied with what he sees, refine the parameters of the object, and launch again. All this in seconds. One can even imagine a cooperative and parallel exploration: by making available a smartphone application where the user can launch a design, with his own parameters, collect the results, ask people to rate them. This could reveal hidden design rules and interesting properties. Several authors developped dedicated softwares to explore mathematical objects. One can cite Ken Stephenson [2] in the field of circle packings, and Phillip Kent [1] for conical, cylindrical and pyramidal anamorphoses.

[1] Möbius transforms were in fact first described by Euler in *Acta Acta academiae scientiarum Petropolitanae*, in 1777 (see [4])

References

[1] Anamorph me! website (with software).
http://www.anamorphosis.com.

[2] Ken Stephenson's website (with circle packing software).
http://www.math.utk.edu/~kens/CirclePack/.

[3] Jurgis Baltrušaitis. *Anamorphoses ou Thaumaturgus Opticus*. Idées et Recherches. Flammarion, 1984.

[4] Claude Bruter. Complex history, Letters to the Editor. *Notices Amer. Math. Soc.*, 54(8):958, 2007.

[5] Francesco De Comité. A General Procedure for the Construction of Mirror Anamorphoses. In George W. Hart and Reza Sarhangi, editors, *Proceedings of Bridges 2010: Mathematics, Music, Art, Architecture, Culture*, pages 231–238, Phoenix, Arizona, 2010. Tessellations Publishing.

[6] Francesco De Comité. A New Kind of Three-Dimensional Anamorphosis. In Reza Sarhangi and Carlo H. Séquin, editors, *Proceedings of Bridges 2011: Mathematics, Music, Art, Architecture, Culture*, pages 33–38, Phoenix, Arizona, 2011. Tessellations Publishing.

[7] Francesco De Comité. Circle Packing Explorations. In George W. Hart and Reza Sarhangi, editors, *Proceedings of Bridges 2013: Mathematics, Music, Art, Architecture, Culture*, pages 399–402, Phoenix, Arizona, 2013. Tessellations Publishing.

[8] R. Fricke and F. Klein. *Vorlesungen über die Theorie der Automorphen Functionen*. Number vol. 1 in Vorlesungen über die theorie der automorphen functionen. B.G. Teubner, 1897.

[9] James Hopkins. James Hopkins' website.
http://www.jameshopkinsworks.com.

[10] Tiffany C. Inglis and Craig S. Kaplan. Circle patterns in gothic architecture. In Robert Bosch, Douglas McKenna, and Reza Sarhangi, editors, *Proceedings of Bridges 2012: Mathematics, Music, Art, Architecture, Culture*, pages 133–140, Phoenix, Arizona, USA, 2012. Tessellations Publishing.

[11] David Mumford, Caroline Series, and David Wright. *Indra's Pearls: the Vision of Felix Klein*. Cambridge Univ. Press, Cambridge, 2002.

[12] Dan Pedoe. *Circles: A Mathematical View (Spectrum)*. The Mathematical Association of America, 2nd edition, 1997.

[13] Kenneth Stephenson. *Introduction to Circle Packing. The Theory of Discrete Analytic Functions*. Cambridge Univ. Press, Cambridge, 2005.

[14] I. Stewart, A.C. Clarke, and N. Lesmoir-Gordon. *The Colours of Infinity: The Beauty and Power of Fractals*. Clear Books, 2004.

[15] William Thurston. The Finite Riemann Mapping Theorem, 1985. Invited talk, Purdue University.

Francesco De Comité
Associate Professor in Computer Science (University of Lille)
CRIStAL
bât. M3 Info
Université des Sciences de Lille
Cité Scientifique
59655 Villeneuve d'Ascq
Francesco.De-Comite@univ-lille1.fr France

INVERTING BEAUTY

Renzo Caddeo, Gregorio Franzoni, Paola Piu

Abstract

In this paper we give a simple application of spherical inversion, the most elementary among the non elementary geometric transformations, and of some of its generalizations.

The principal motivation was an attempt to increase the interest for mathematics in high school students by proposing an easy but mathematically rigorous technique for creating new images, new shapes and, by means of 3D printing, new nice material objects. Also in order to put once again in evidence the possibility that mathematics can have something in common with Nature and the Arts.

Amongst the generalizations of inversion (see [Bl], [Hi], [Ep], [S1]), we find ideally more close to our point of view the hyperbolic inversion due to G. V. Schiaparelli[1], an important Italian astronomer not as much known as a geometer, who in [S2], in 1898, tried to represent organic forms and the change from one species to another through geometry (see, e.g., [Gi-Gu]).

1 Introduction

Nature offers our eyes every day extraordinarily beautiful forms, that look always the same and always new, but that never fail to amaze us. An example is represented by flowers: the pleasure we get from them is one of the most intense. Maybe is this one of the reasons why artists often give in to the temptation to reproduce them, sometimes emulating, sometimes interpreting nature, that is, deforming their shape.

Everyone has their own preferences. Many have predilection for roses, but some find calla-lilies (or arum lilies) more fascinating, because of their slender elegance, which firmly soars spiraling upward. It's not hard to find callas in vases and gardens, or in paintings. A bouquet of callas, as that in Fig.1, is an

[1]Giovanni Virginio Schiaparelli (1835-1910), astronomer and historian of science, senator of Kingdom of Italy, Bruce and Royal Astronomical Society gold medalist, discovered groups of straight lines (*canals*) on Mars, raising doubts on existence of life on that planet, and gave an explication of shooting stars as residues of comets. The relationship between his hyperbolic transformation and standard inversion was observed by Luigi Cremona.

Fig. 1 Bouquet of calla-lilies

interesting subject to several painters, who may have been attracted by their geometric profile and their nearly evanescence.

Drawings and paintings with callas as subject can be easily found on the net, taken from museums and from more or less important art galleries.

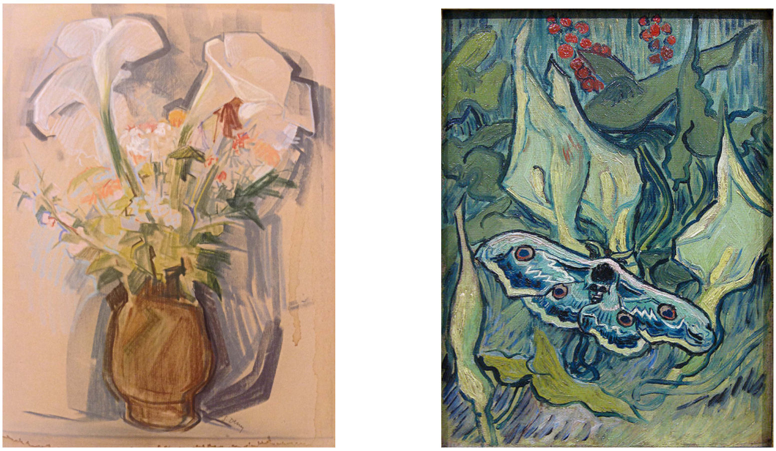


Fig. 2 (a) Calla-lilies (b) Great Peacock Moth

In Fig.2a and Fig.2b we have reproduced a watercolour painting by Stanis Dessy, a Sardinian artist (1900-1986), and a Van Gogh's picture.

It is interesting to know that also in mathematics one can find a *calla*: the extremely elegant and beautiful surface which bears her discoverer's name, Ulisse Dini, (see [Di]). It can be drawn with *Mathematica* (see for example [Ca-Gr]) by using the parametrization given by the map

$$X(u,v)=\left\{b\sin u\sin v, b\sin u\cos v, b\left(\cos u+\ln\left[\tan\left(\frac{u}{2}\right)\right]\right)+cv\right\}. \quad (10.1)$$

For $a=32$, $b=5$, $u\in[-3\pi,-0.5]$ and $v\in[0.02,\pi/2-0.02]$, we get

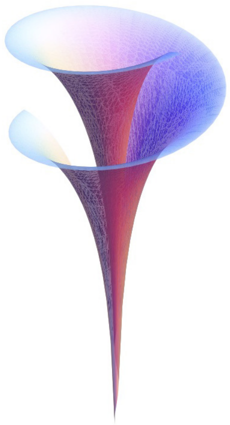


Fig. 3 Dini's Surface

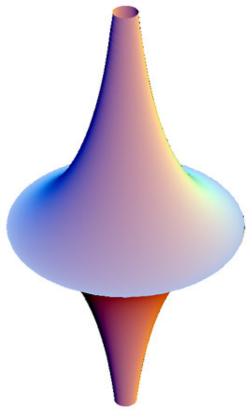


Fig. 4 Pseudosphere

Dini's surface can be obtained through an isometric deformation from the Eugenio Beltrami's *Pseudosphere* [2], which is parametrized by

$$X(u,v)=\left\{\sin u\sin v, \sin u\cos v, \cos u+\ln\left[\tan\left(\frac{u}{2}\right)\right]\right\}.$$

Therefore these two surfaces, even though it does not appear evident at first glance, share the property of being curved in the same way in every point. More precisely, they have Gauss curvature K constant and equal to -1.

One may wonder whether it is possible to create by means of mathematical tools a composition resembling any of those in Fig.1 and in Fig.2. In order to draw a bouquet from the Dini's surface, it is necessary to find a way of *bending* it in a proper way, to obtain a visually pleasing composition.

It is important to observe that this can be done through a *spherical inversion*. This is a non elementary (i.e. non linear) geometric transformation, in fact one of the simplest, besides rigid motions (the congruences) of Euclidean geometry.

In the following sections we will briefly recall the notion of inversion, some of its well known properties and some generalizations, not as much well known. But first of all we show what our bunch of "flowers" looks like:

[2] Section 2 of Livia Giacardi's interesting paper [Gi] of 2013 ESMA proceedings is devoted to the Beltrami's cardboard model of this pseudo spherical surface.

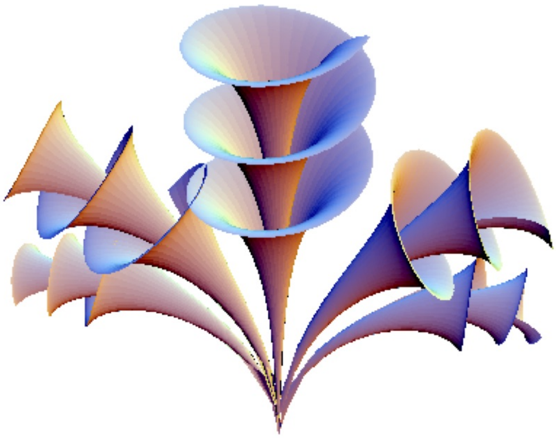
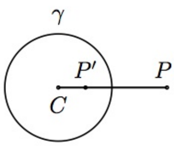


Fig. 5 A Dini bouquet

This composition has been realized by putting together one Dini's surface and four of its inverses, obtained by inverting Dini's surface with respect to the two spheres of radius 150 and centered in the points $(-0.63, 150, -162)$, and $(-0.63, -150, -162)$, and to the two spheres of radius 280 with centers in $(-0.63, 280, -162)$ and $(-0.63, -280, -162)$.

2 Inversion in Circles

There are several kinds of inversion (see [Hi]); among them, the inversion with respect to a circle is the simplest one. In Sect.6 we shall mention some other inversions; let us begin with the circular inversion.



Let γ be a circle of radius r and center C. By definition, for any point $P \neq C$ in the plane of γ, the **inverse** P' of P with respect to γ is the unique point P' on the half-line containing C and P such that

$$\overline{CP} \cdot \overline{CP'} = r^2. \tag{10.2}$$

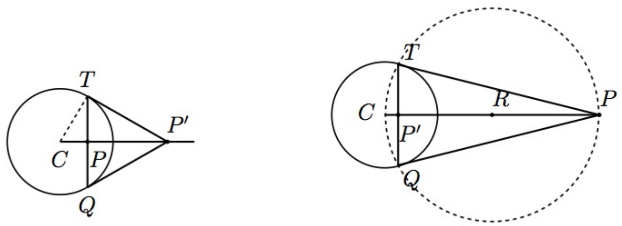
The number r^2 is the *power* of inversion.

From the definition it is easy to determine the mutual position of P and P' with respect to γ:

1. $P = P'$ if and only if P lies on the circle γ.
2. If P is inside γ, then P' is outside γ, and P' is inside γ if P is outside.
3. $(P')' = P$ (that is, the inverse of the inverse of P is P).

Moreover, there are very simple geometric constructions to find the inverse P' of $P \neq C$. Besides the trivial case, when P belongs to γ, we have two cases:

1. **The point P is inside γ.** Let TQ be the chord of γ through P perpendicular to $\overrightarrow{CP}$. Then the inverse P' of P is the point of intersection of the tangents to γ at T and Q.
2. **The point P is outside γ.** Let R be the midpoint of the segment CP, and σ the circle with center R and radius $\overline{CR} = \overline{PR}$. Then σ intersects γ in T and Q, PT and PQ are tangent to γ, and the inverse P' of P is the intersection of TQ and CP.

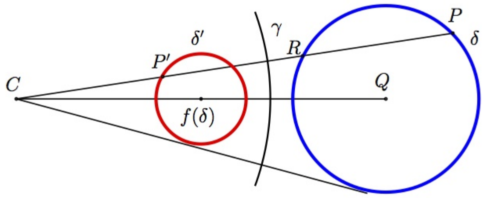


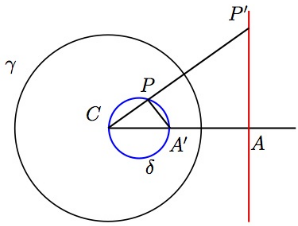
Once we have learnt how to find the inverse of a given point, it is interesting to see how are the inverses of sets of points. If the figure to invert is a circle, the result, simple and surprising, is a Steiner's theorem that in [Pa], p.178, is called the **fundamental theorem of inversion**:

Theorem 2.1 *The inverse of a circle is a circle.*

More precisely one has the following two cases:

1. Let γ be a circle of radius r and center C, δ a circle of radius s and center Q. Assume C outside δ and let k be the power of C with respect to δ. Let f the dilation with center C and ratio $\lambda = r^2/k$. Then the image δ' of δ under inversion in γ is the circle of radius $\lambda \cdot s$ whose center is the image $f(Q)$ of Q.
2. Let δ be a circle passing through the center C of a circle γ. The image of δ minus C under inversion in γ is a line ℓ not through of the center C; the line ℓ is parallel to the tangent to δ at C.





It is also well worth considering inversions of other conic sections. We must here remark that, to represent with the program *Mathematica* the inverse of a parametrized curve, it is convenient to translate condition (10.2) in the formulas relating the coordinates of a point P and those of its inverse. These formulas are obtained by observing that, when the point P describes the curve α, the inverse curve α', is drawn by the point P' given by

$$P' = C + \frac{r^2(P-C)}{||P-C||^2}. \tag{10.3}$$

Here we have denoted by $||P-C||$ the length of the segment $\overline{CP}$.

Let us show (in red) some inverse curves of conics. Inversion of a parabola with respect to circles centered at its vertex and its focus gives, respectively, a *cissoid of Diocles* and a *cardioid*:

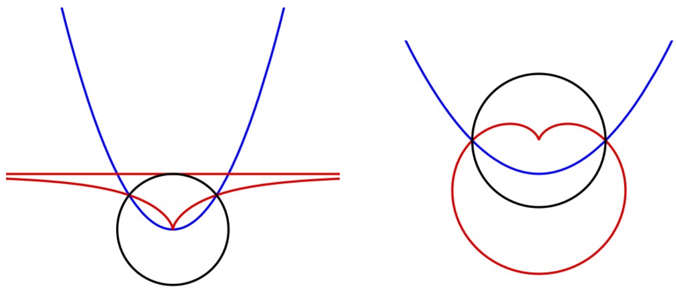


Fig. 6 A parabola with its inverse with respect to circles centered in its vertex (*left*) and in its focus (*right*)

For an ellipse, taking the circle of inversion centered in a vertex, in the center and in a focus of the ellipse, we obtain respectively a *witch of Agnesi*, a *lemniscate of Booth* and a *limaçon of Pascal*

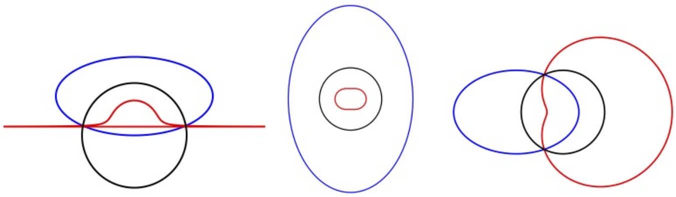


Fig. 7 An ellipse with its inverses with respect to circles centered in one of its vertices (*left*), in the center (*middle*) and in one of its foci (*right*)

When the conic we want to invert is a hyperbola and the circles of inversion are chosen as for the above ellipse, we get a *strophoid*, a *lemniscate of Bernoulli* and a *limaçon of Pascal*, respectively.

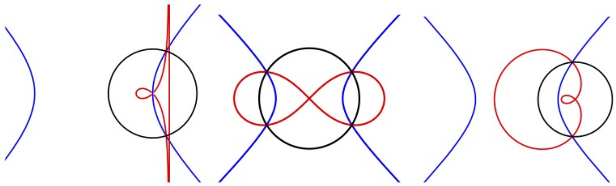


Fig. 8 A hyperbola with its inverses with respect to circles centered in one of its vertices (*left*), in its center (*middle*) and in one of its foci (*right*)

But it is also interesting to see in what way the inversion deforms triangles and squares.

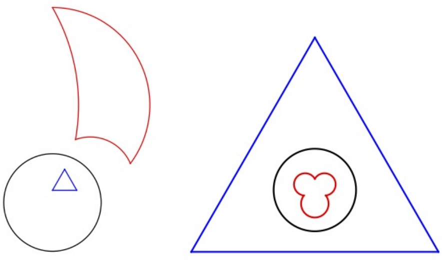


Fig. 9 Inverses of a triangle contained in (*left*) and containing (*right*) the circle of inversion

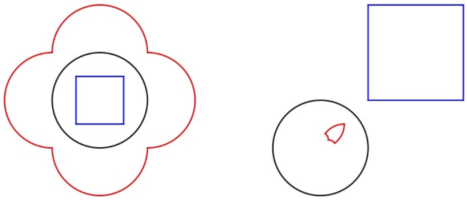


Fig. 10 Inverses of a square with respect to a circle inside the square, and to a circle outside the square

3 Inversions in three dimensions

Let us now consider the inversion with respect to a sphere, to explain the way Fig.5 was obtained.

Conditions (10.2) and (10.3) do not change when we want to invert a point $P \neq O$ in ordinary space with respect to a sphere centered in O and of radius r. But now, besides the curves, we can invert planes, spheres, quadrics and more complicated surfaces, as it can be seen in [Ca-Gr].

For example, in Fig.11 are represented the Möbius strip

$$X(u,v) = \left\{\cos u + v\cos\frac{u}{2}\cos u, \sin u + v\cos\frac{u}{2}\sin u, v\sin\frac{u}{2}\right\}$$

and its inverse with respect to the sphere of radius 2 centered in the origin:

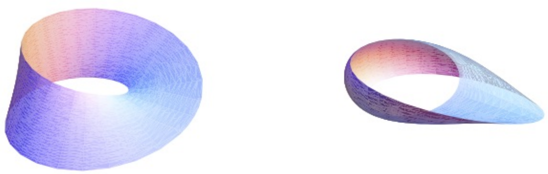


Fig. 11 A Möbius strip with its inverse

In Fig.12 we can see a torus and, on the right, one of the famous Dupin cyclides, which held also J. C. Maxwell's interest (see [Mx1]). This cyclide can be obtained by inverting the torus

$$X(u,v) = \{\cos u(8+3\cos v), \sin u(8+3\cos v), 3\sin v\}$$

with respect to the sphere centered in the point $C = (0,2,0)$ and of radius 2.

Now we draw in Fig.13 two of the surfaces we need to compose the image in Fig.5.

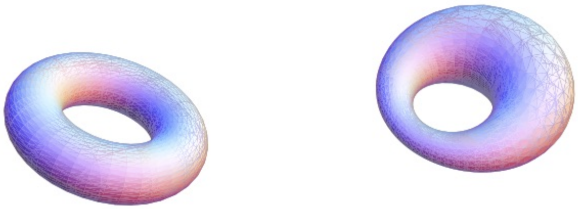


Fig. 12 A torus with its inverse

The inverses of the Dini's surface given by parametrization (10.1) with respect to the sphere centered in $(-0.63, 150, -162)$ and of radius 150 and with respect to the sphere centered in $(-0.63, 280, -162)$ and of radius 280 are:

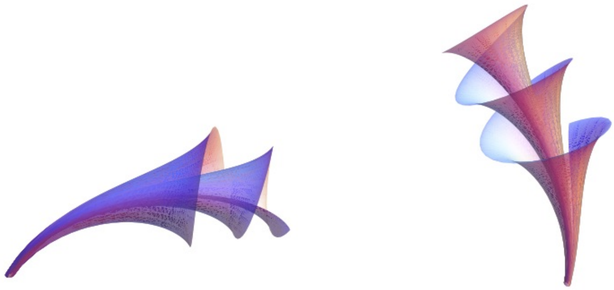


Fig. 13

And finally we are able to obtain the bouquet in Fig.5. We just put together the surfaces in Figures 3 and 13 and the two other inverses that we get by inverting the Dini's surface with respect to the sphere centered in $(-0.63, -150, -162)$ of radius 150, and to the sphere centered in $(-0.63, -280, -162)$ of radius 280.

4 Digression. 3*D* Printing of the Dini bouquet

Probably the reader knows what 3D printing is. During the last 4 - 5 years it sticked out from technical reviews and specialist environment to come to mass reviews and TV. In a certain sense, 3D printing "brings to real life" solid objects who live in the virtual worlds created by computers. Thanks to it we can realize and keep in our hand a very precise copy of the Dini surface bouquet. Those who do not know enough about 3D printing techniques will find some details further on this paragraph. The picture below shows a virtual model of the bouquet (on the left) and a photography of the corresponding physical model of it.

Let us spend some words more about this technique. First of all, any application of these or other surfaces in physical world bases on the realization of material models of such objects. By means of scientific, design or engineering software it is possible to get 3D models, i.e. models that exist in a virtual three dimensional space in the memory of a computer and of which we can see in perspective some projections and animations giving us the impression of being watching real objects.

Techniques of *rapid prototyping* and *3D printing*, born in the end of the eighities of last century, allow to carry out the next step, that is to take mathematic surfaces to the concrete, real world in the form of tangible objects, which we can hold in our hands, rotate, and observe from different angles, to get a precise idea of their geometrical and topological properties. The mentioned terms indicate a series of techniques widely used to build conceptual and functional prototypes in several industrial fields, like automotive, electric household appliances, toys, jewels and medical fields, with not negligible applications in artistic, cultural and archeological areas. The main innovation introduced by these techniques (in the following we will use just the 3*D printing* term, which generally indicates medium to low cost systems that do not require a specific technical competence to users, while the *rapid prototyping* term is associated to the industrial/professional systems; the bases of the operating principles are the same) is that they make realizable every kind of shape, no matter how complicated they are, with the sole condition that they represent real solid objects, i.e. not impossible figures. Shapes can be complex, can have back drafts, undercuts, inner canals of cavities, features that make them impossible to be realized by means of more traditional techniques such as lathe or CNC cutter (for example: a car's engine block, including the duct for the liquid coolant, or the accurate reproduction of an human skull). This is made possible thanks to the working method of *3D printers*. They decompose the 3D model to be realized –

usually a closed triangle mesh – into a collection of plane parallel sections. Then they execute the realization of every layer, in thickness variable between 0.05 and 0.5 mm depending on technology, until they produce a concrete part that corresponds to the virtual object. Materials used vary depending on technology: photopolymers, ABS resin, nylon, plaster, ceramic, metal.

Fig. 14 Photos of 3D prints of models of surfaces; From left: part of the Klein bottle in Lawson's version; Moebius band with circular boundary; Boy surface according to F. Apéry's parametrization (see [Ap]).

So, what do we have to do if we have the equations of a surface and we want a physical model of it? Unfortunately, the operation is not straightforward. In fact, the input needed for an RP system is a watertight polyhedral mesh which represents a real object. In the main applications of 3D printing (design of new products in industrial field) the solid model is produced by *solid modeling* software, specifically conceived to give outputs ready to use with 3D printers, or by reverse engineering techniques. There is a survey of scientific software which allow the representation and visualization of surfaces starting from their parametrization (equation), like *Mathematica*, *Maple*, *MatLab*, *MathCad*, but such representations are not usable on a 3D printer. It is necessary to shift from the bidimensional exhibits needed in visual/graphic environments to volume-including shells suitable to 3D printing environment. This idea can be transferred into a series of mathematical operational steps, whereof starting with a surface parametrization we end up having a closed mesh directly usable by a 3D printer, which effectively represents our surface. It is relatively simple to obtain a printable solid model for regular surfaces without multiple loci by using the basic tools of differential geometry: it suffices to define some parallel and normal surfaces to the given one to construct a solid shell around it. Otherwise, in presence of self-intersections and/or singularities (and these are often the most interesting cases) we need to solve nontrivial problems which involve more differential geometry and computational geometry and which request skills transversal to both the mentioned scientific fields.

We finish this short journey into the world of $3D$ printing saying that in the last decade 3D printers performances have been constantly increasing while their price has decreased. In a close future, that probably has already begun, it will be normal to have a $3D$ printer connected to our PC just like today everyone has a inkjet or laser printer. Today (2013) it is already possible to buy a small $3D$ printer kit, with basic performances, at a price as low as 600 dollars.

5 Notes on circular inversion

Historically, the interest for transformations of the entire plane and for the properties that are invariant with respect to them, to solve geometrical problems, comes from the development of projective geometry of the XIX century, mainly due to Gaspard Monge (1746-1818) and to Jean-Victor Poncelet (1788-1867).

As regards the inversion, to our knowledge the most authoritative references are, in chronological order, the F. Bützberger pamphlet [Bü], reviewed by Arnold Emch in the Bulletin of the American Mathematical Society, vol. 20 (1914), pp. 412 - 415, and the very interesting Boyd C. Patterson's paper [Pa] on the origins of the geometric principle of inversion. Another useful reference for circular inversion and its generalizations, including those mentioned in the last two sections of this paper, is [Ca].

Let us follow the more important stages of this route from the beginning, in chronological order.

The first paper containing *in nuce* the idea of inversion appears in 1600, when François Viète gives a solution of the tenth Apollonius problem [3]. On pages from 5 to 9 of [Vi], Viète presents a solution by using the center of similitude of two circles.

After more than two centuries, the belgian mathematicians (and good friends) Germinal Pierre Dandelin [Da] and Adolphe Quételet [Qu], also known for the so-called *Belgian Theorems* (see, e.g., [Hu]), arrive, independently one from the other, to the principle of inversion when studying the properties of the *focale* of conic sections (see, e.g., [Pa], p. 156) by means of stereographic projection. In particular, Quételet deduces the relation $rr' = R^2$ between *radii vectores* that are reciprocal with respect to a circle of radius R, and also the explicit analytic formulas of the circular inversion.

But the first who gives the precise definition of inversion and establishes and applies inversion is Jakob Steiner, during his researches on the geometry of

[3] PROBLEMA X: *Datis tribus circulis, describere quartum circulum quem illi contingant.* (*To draw a circle that touches three given circles in a plane.*)

the circle and the sphere through the theory of similar figures (see [Bü]). Later, in 1847, Joseph Liouville will give to the inversion the name of *transformation par rayons vecteurs réciproques*. Again from [Bü] it comes out that Steiner was interested in this study in the attempt to solve the following problem: given in a plane three circles, to find the locus of a point whose polars with respect to three circles pass through a point.

In 1831, Julius Plücker (see [Pl]) finds inversion within the theory of poles and polars (in the next section we shall recall the construction of poles and polars of a conic), while in 1832 Ludwig Immanuel Magnus, in [Ma], discovers inversion as a particular case of a bijective map between two planes.

In 1836, Giusto Bellavitis, informed, even if partially, about the results obtained by Dandelin and Quételet, realizes that inversion is a very useful instrument that, combined with other geometric tools, could lead to new theorems. His memoir [Bl] contains an elementary and very complete exposition of various geometric transformations such as those of similarity, projection, inversion, reciprocal polars, homology.

In 1842 there are two more simultaneous and independent discoveries of a *new principle* that is nothing but the inversion. The mathematicians involved are John William Stubbs and John Kells Ingram, both at Trinity College in Dublin; three of their papers on the subject even appear in the same volume of the *Transactions of the Dublin Philosophical Society* (see [St1], [St2], [In1] and [In1]).

In 1845, William Thomson (Lord Kelvin) communicates to Liouville his "principle of reciprocal points", a method he found useful to solve a certain problem in electricity. Louville develops the analytic theory of the Thomson's transformation that in this occasion he calls *transformation by reciprocal radii*.

The last we have to mention is August Ferdinand Möbius, that in 1855 undertakes in [Mö] a systematic study of inversion.

6 Generalizations

A quite reasonable question is whether there exist other similar, simple, geometric, useful constructions, that can also give rise to new shapes. In 1838, Giusto Bellavitis of Padua proposed in [Bl] a very natural generalization of the circular inversion by taking any conic instead of a fixed circle, and allowing the center of the inversion to be placed anywhere, and not only in a special position as, for example, in a center of symmetry.

Such a generalization can be obtained by considering that, as for the circle, for any conic there is a canonic, geometric way to find the point P' inverse of a point P. This happens thanks to the fact that a conic determines in its plane a correspondence between points P, the *poles*, and straight lines p, their *polars*.

For example, given an ellipse σ, if P is a point outside σ, its polar p is the straight line through the points Q and T where the two tangents to σ from P touch the conic.

When P is a point of σ, there is only one tangent to σ at P, and therefore this tangent coincides with the polar p of P.

If P is inside σ, any two straight lines r_1, r_2, $r_1 \neq r_2$, passing through P, intersect σ in four points A_1, B_1, A_2, B_2. Let Q and T be the intersections of the lines a_1, b_1 and a_2, b_2, respectively tangent to σ at the points A_1, B_1 and A_2, B_2. Then the polar p of P is the straight line through Q and T. The following figures illustrate the first and the third case.

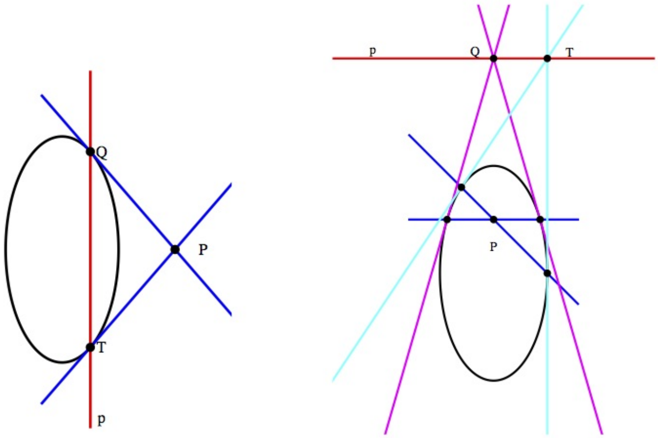


Fig. 15 Polars (red) of a point P outside or inside an ellipse

Therefore, according to Ludwig Immanuel Magnus and Julius Plücker (see [Ma] and [Pl]), we can define the inversion with respect to a conic σ in the following way.

Definition 6.1 *Let A be a point fixed as origin in the plane of σ. Then the inverse P' of a point P of this plane is the intersection between the polar of P with respect to σ and the straight line through P and the origin A.*

In Fig.16 are represented an ellipse and the construction of P' when the point P is outside (left) and inside (right) an ellipse:

If P belongs to σ, its polar line is the tangent to σ at P and the straight line through P and A intersects this tangent at P. Thus $P = P'$.

Except for the origin A and the two points (real or imaginary) where the tangents from A to σ touch σ (these are the three *principal points*), every point P has only one inverse P'.

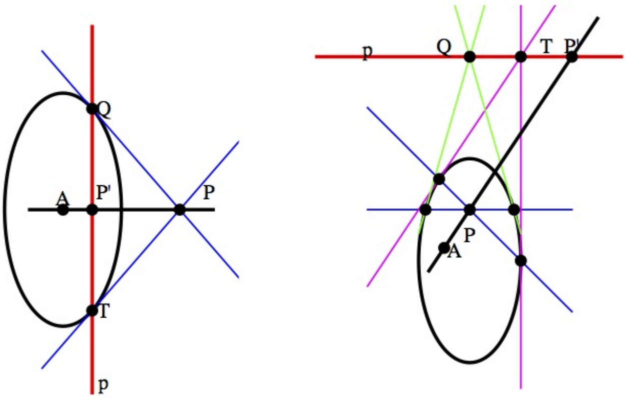


Fig. 16

It is not difficult to foresee that the study of the inversions with respect to the different conics and to various origins can be even rather complicated. This work has been achieved in 1865 by T. A. Hirst in the paper *On the quadric inversion of plane curves* [Hi][4], where one can find an ample and detailed essay on the subject. Here we shall only mention and illustrate by drawings the more interesting and useful of the special cases corresponding to particular choices of the conic and of the origin. In the following, the *fundamental conic* is the conic with respect to which we invert points.

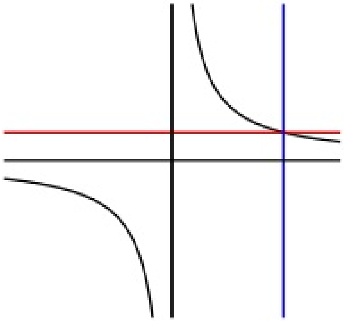
(I) **The fundamental conic is a hyperbola with its centre at the origin A.**

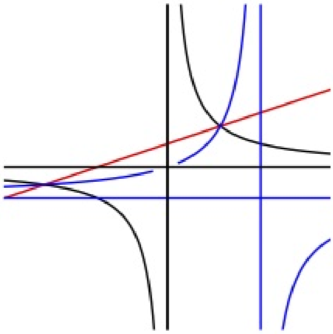
The inverse of every straight line parallel to one of its asymptotes is a straight line parallel to the other asymptote, and the two straight lines intersect on the fundamental hyperbola.

The inverse of every other straight line is a hyperbola passing through the origin, and having its asymptotes parallel to those of the fundamental hyperbola.

The inverse of any hyperbola which does not pass through the origin, but has its asymptotes parallel to those of the fundamental conic, is an hyperbola possessing the same properties; and if the hyperbola we want

[4] An Italian version of this paper, published in *Annali di Matematica Pura e Applicata,* Serie 1, Dicembre 1865, vol. 7,1, pp. 49-65, with title "Sulla Inversione quadrica delle curve piane", is due to Luigi Cremona. Here is how the paper is introduced: *We consider a good and useful thing to bring this important and very elegant work of our friend, Mr. Hirst, to the knowledge of the readers of the Annali.* (*Stimiamo cosa buona e utile il far conoscere ai lettori degli Annali questo importante ed elegantissimo lavoro del nostro amico, il Sig. Hirst.*) (Luigi Cremona)





to invert has centre at the origin, its inverse will have centre in the origin as well.

(Ia) **The conic is an equilateral hyperbola.**

Choosing again the origin at the centre of the hyperbola, the method of inversion becomes identical with the *hyperbolic transformation* investigated by Giovanni Virginio Schiaparelli, in his interesting memoir *Sulla trasformazione geometrica delle figure* (see [S1]).

(II) **The fundamental conic is an ellipse and the origin is at its centre.**

The inverse of every straight line in the plane will be an ellipse passing through the origin, and at the same time similar, as well as similarly placed to the fundamental ellipse (that is, in the two ellipses, the axes of symmetry corresponding in the similarity are parallel).

Every ellipse not passing through the origin, but similar and similarly placed to the fundamental one, has for its inverse an ellipse with the same properties; and should the primitive be likewise concentric with the fundamental ellipse, so also will be the inverse:

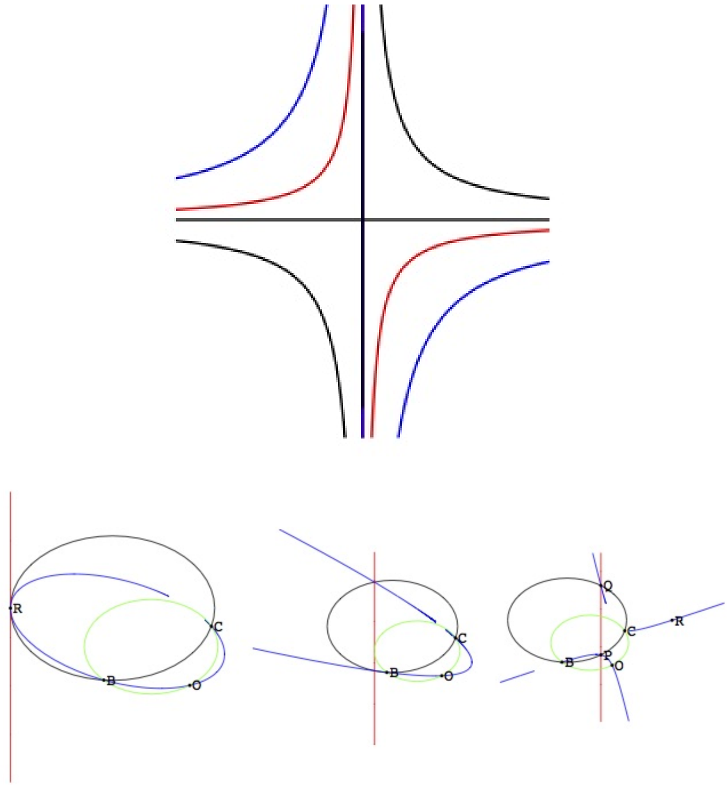


Fig. 17 Inversion of three parallel straight lines

7 Schiaparelli's hyperbolic inversion of some surfaces

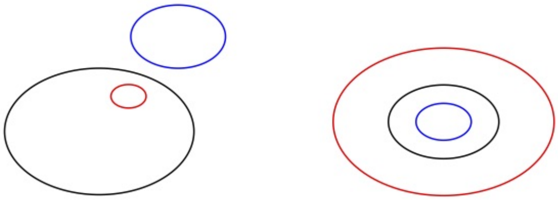
In this section, we illustrate by some examples how the Schiaparelli hyperbolic inversion transforms figures and bends surfaces. Other examples can be found in [Pe].

To obtain a generalization of the spherical inversion, Schiaparelli substitutes the sphere with a quadric (see [S1]). Here we only consider the particular case when the quadric is a precise hyperboloid of two sheets.

- The circular cylinder of parametrical equations

$$\begin{cases} x(u,v) & = a\cos u \\ y(u,v) & = a\sin u \\ z(u,v) & = v \end{cases}$$

with respect to the hyperboloid of two sheets of equation $xy + xz + yz = 1$



has as inverse

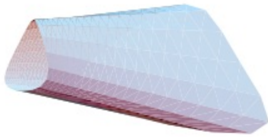


Fig. 18 Hyperbolic inverse of a cylinder

- The inverse of the pseudosphere of parametric equations

$$\begin{cases} x(u,v) &= a\cos u \sin v \\ y(u,v) &= a\cos u \cos v \\ z(u,v) &= a\cos v + a\ln(\tan(\frac{v}{2})) \end{cases}$$

with respect to the same hyperboloid is

Fig. 19 Hyperbolic inverse of a pseudosphere

- Next we consider the Dini's surface of parametrical equations

$$\begin{cases} x(u,v) &= a\cos u \sin v \\ y(u,v) &= a\cos u \cos v \\ z(u,v) &= a\cos v + a\ln(\tan(\frac{v}{2})) + bu \end{cases}$$

Its inverse with respect to the same hyperboloid of two sheets is

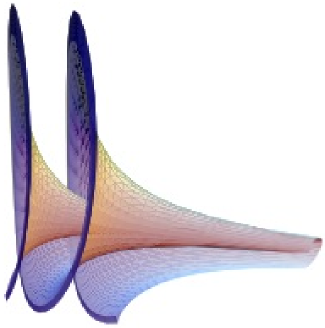


Fig. 20 Hyperbolic inverse of a Dini's surface

- Finally we apply the same inversion to the tori of parametric equations

$$X_1(u,v) = \{\cos u(8+3\cos v) - 12, \sin u(8+3\cos v) - 12, 3\sin v + 20\}$$

and

$$X_2(u,v) = \{\cos u(8+3\cos v) - 20, \sin u(8+3\cos v) - 20, 3\sin v - 20\}.$$

From the left to the right, the corresponding inverse surfaces are

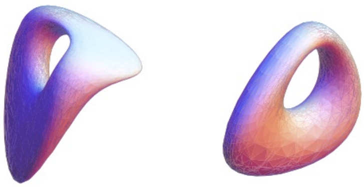


Fig. 21 Hyperbolic inverse of a Dini's surface

Remark A modern analytic treatment of inversion with respect to a general quadric of a n-dimensional vector space, endowed with a non-degenerated symmetric bilinear form, can be found in the 1983 D.B.A. Epstein's lecture notes [Ep].

References

[Ap] Apéry F., "An algebraic halfway model for the eversion of the sphere", *Tohoku Mathematical Journal*, 44, 1992.

[Bl] Bellavitis G., “Teoria delle figure inverse, e loro uso nella Geometria elementare”, *Ann. Lomb. Veneto*, 6, 1936, 136-141.

[Bü] Bützberger F., *Über bizentrische Polygone, Steinersche Kreis-und Kugel-reinhen und Erfindung der Inversion*, Teubner, Leipzig, 1913.

[Ca-Gr] Caddeo R., Gray A., *Lezioni di geometria differenziale su curve e superfici*, volumes 1 and 2, CUEC, Cagliari, 2000 and 2001.

[Ca] Cassina U., *Su alcune trasformazioni quadratiche*, Enciclopedia delle Matematiche Elementari, vol. II, Parte I, Hoepli, 1936, 464-476.

[Da] Dandelin G. P., “Mémoire sur quelques propriétés remarquables de la focale parabolique”, Nouveaux Mémoires de l’Académie Royale des Sciences et Belles-Lettres de Bruxelles, t. II (1822), 171-202.

[Di] Dini U., “Sopra alcuni punti della teoria delle superfici applicabili”, *Annali di Matematica Pura e Applicata*, (1) 7, 1865, 25-47.

[Ep] Epstein D.B.A., “Isomorfismi conformi e geometria iperbolica”, *Quaderno del CNR*, Levrotto e Bella, Torino, 1983.

[Gi] Giacardi L., “Models in Mathematics teaching in Italy (1850-1950)” in *Proceedings of the Second ESMA Conference*, held in Cagliari, September 2013 (this volume).

[Gi-Gu] Giorello G., Guzzardi L., “G. V. Schiaparelli: from scientific observations to scientific imagination”, *Mem. S. A. It.* vol. 82, 219, SAIt 2011.

[Hi] Hirst T.A., “Sull’inversione quadrica delle curve piane”, *Annali di matematica pura ed applicata*, 1865.

[Hu] Hubaut X., *Les Théorèmes Belges*, http://xavier.hubaut.info/coursmath/2de/belges.htm

[In1] Ingram J. K., “Geometrical Properties of Certain Surfaces”, *Transactions of the Dublin Philosophical Society*, vol. I, 1842-43, 58-.

[In2] Ingram J. K., “On the Properties of Inverse Curves and Surfaces”, *Transactions of the Dublin Philosophical Society*, vol. I, 1842-43, 159-.

[Li] Liouville J., “Note au sujet de l’article précédent”, *J. Math. Pures et Appl.* (1) 12, 1847, 265-290.

[Lr1] Loria G., *Curve piane speciali algebriche e trascendenti*, Ulrico Hoepli, Milano, 1930.

[Lr2] Loria G., *Storia delle Matematiche*, vol. II, Sten, Torino, 1933.

[Ma] Magnus L.J., “Nouvelle méthode pour découvrir des théorèmes de géométrie”, *J. Reine Angew. Math* (Crelle’s Journal), Bd 8 (1832), 51-63.

[Mx1] Maxwell J.C., “On the cyclide”, *J. of Pure and Applied Math.* 34 (1867), 144-159.

[Mx2] Maxwell J.C., *Traité d’éctricité et de magnétisme*, Tome 1, Gauthiers-Villars, Paris, 1885.

[Mö] Möbius A.F., “Theorie der Kreisverwandschaften in rein geometrischer Darstellung”, *Abh. Königl. Sachs. Gesellschaft Wiss., Math. Phys. Kl.* 2, 529 (1855).

[Pa] Patterson B.C., “The origins of the Geometric Principle of Inversion”, *Isis*, Vol. 19, No. 1 (Apr.,1933), 154 - 180.

[Pe] Perra L.B.D., *Inversioni*, Degree thesis. Università degli Studi di Cagliari, 1999.

[Pl] Plücker J., “Analytisch-geometrische Aphorismen”, *J. Reine Angew. Math* (Crelle’s Journal), Sect V, vol. XI, (1834), 219-225.

[Qu] Quételet A., “Mémoire sur une nouvelle manière de considérer les caustiques, produites, soit par réflexion soit par réfraction”, *Nouveaux Mémoires de l’Académie Royale des Sciences et Belles-Lettres de Bruxelles*, t. III (1826), 89-140.

[S1] Schiaparelli G.V., “Trasformazione geometrica delle figure piane”, *Memoria Accademia di Torino*, serie II, tomo XXI, 1861.

[S2] Schiaparelli G.V., *Studio comparativo tra le forme organiche naturali e le forme geometriche pure*, Hoepli, Milano,1898.

[St1] Stubbs J.W. “On the Application of a New Principle to the Geometry of Curves and Curve Surfaces”, *Transactions of the Dublin Philosophical Society*, vol. I, 1842-43, 145-.

[St2] Stubbs J.W. “On the application of a new Method to the Geometry of Curves and Curve Surfaces”, *Philosophical Magazine*, vol. XXIII, 1843, 338-347.

[T1] Thomson W., “Extraits d’une lettre à M. Liouville”, *J. Math. Pures et Appl.*, (1) 10, 1845, 364-367.

[T2] Thomson W., “Extraits de deux lettres adressées à M. Liouville”, *J. Math. Pures et Appl.* (1) 12,1847, 256-264.

[Vi] Viète F., *Apollonius Gallus*, David Le Clerc, Paris, 1600.

Renzo Caddeo
Professor (Università degli Studi di Cagliari)
Dipartimento di Matematica e Informatica
via Ospedale, 72
09124 Cagliari
Italia
caddeo@unica.it

Gregorio Franzoni
Teacher of mathematics in secondary school and
university lecturer (Università degli Studi di Cagliari)
Dipartimento di Matematica e Informatica
via Ospedale, 72
09124 Cagliari
Italia
gregoriofranzoni@gmail.com

Paola Piu
Assistant Professor (Università degli Studi di Cagliari) and
co-ordinator for Sardinia of the Italian Ministry of Education program “Piano Lauree Scientifiche”
Dipartimento di Matematica e Informatica
via Ospedale, 72
09124 Cagliari
Italia
piu@unica.it

MINKOWSKI OPERATIONS IN SHAPE MODELLING

Daniela Velichová

Abstract

Visualization of geometric structures obtained from Minkowski sums and Minkowski products of point sets in space E^n leads to views of surface patches demonstrating certain unusual artistic and aesthetic values and can be regarded as objects that stimulate our brains to develop higher dimensional imagination.

1 Introduction

Operation of Minkowski sum of two point sets was introduced by Hermann Minkowski in 1903 during his close cooperation with David Hilbert at the university in Göttingen. Nowadays this set operation has been re-introduced in connection to finding geometric algorithms for description of specific geometric problems dealing with layout optimization, planning trajectory of a robot rigid motion in the working space avoiding obstacles, in offsetting and dense packing, for determination of equidistant manifolds and for shape modeling and morphing purposes in computer graphics, e.g. [1], [2].
The most common interpretation of Minkowski sum of two point sets is by means of vector sum of the position vectors of all points in the given sets.
Let us consider infinite sets of points A and B, which are smooth manifolds in E^n determined by vector maps

$$A: \ {}^1\mathbf{r}(u) = ({}^1r_1(u), {}^1r_2(u), ..., {}^1r_n(u)), \ \ u \in [\alpha, \alpha']$$
$$B: \ {}^2\mathbf{r}(v) = ({}^2r_1(v), {}^2r_2(v), ..., {}^2r_n(v), \ \ v \in [\beta, \beta']$$

Minkowski sum C of curves A and B is defined by

$$C: \mathbf{r}(u,v) = ({}^1r_1(u) + {}^2r_1(v), {}^1r_2(u) + {}^2r_2(v), ..., {}^1r_n(u) + {}^2r_n(v)),$$
$$(u,v) \in [\alpha, \alpha'] \times [\beta, \beta']$$

Minkowski sum of parabolic arc and ellipse, both equally parameterised for $u = v$, is illustrated in Figure 1, on the left, while Minkowski sum of parabolic arc and lemniscate of Bernoulli on the right in Figure 1.

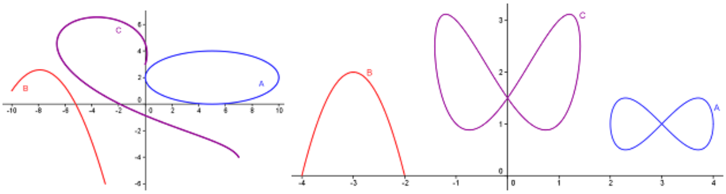


Fig. 1 Minkowski sum of equally parametrized curve segments

Some illustrations of surface patches that are generated as Minkowski sum of two curve segments in the space $\mathbf{E}^3$ parameterised for different parameters u, v are presented in Figure 2. The basic curve segments are planar curves located in two mutually perpendicular planes, and from the left to the right they are the following: two ellipses (1), two parabolic arcs (2), versiére and shamrock curve (3), and finally asteroid and chain curve (4). Surface patches that are Minkowski sums of the respective pairs of curves are presented in their orthographic axonometric views. Equations of curves are the following

$$A: {}^1\mathbf{r}(u) = ({}^1a\cos u,\ {}^1b\sin u, 0),$$
$$B: {}^2\mathbf{r}(v) = ({}^2a\cos v, 0, {}^2b\sin v),\ \ du, v \in [0, 2\pi]$$

$$A: {}^1\mathbf{r}(u) = ({}^1au,\ {}^1bu^2 + {}^1cu + {}^1d, 0),$$
$$B: {}^2\mathbf{r}(v) = ({}^2av, 0, {}^2bv^2 + {}^2cv + {}^2d),\ \ u, v \in [0, 1]$$

$$A: {}^1\mathbf{r}(u) = ({}^1a(2u-1), 0, {}^1b/({}^1c + ({}^1a(2u-1))^2),\ \ u \in [0, 1]$$
$$B: {}^2\mathbf{r}(v) = (0, {}^2a\cos v\sin^2 v, 0, {}^2a\cos^2 v\sin v),\ \ v \in [0, 2\pi]$$

$$A: {}^1\mathbf{r}(u) = (0, {}^1a(\cos 3u + 3(\cos u - 1)), -{}^1a(\sin 3u + 3\sin 3u)),\ \ u \in [0, 2\pi]$$
$$B: {}^2\mathbf{r}(v) = ({}^2a(2v-1), 0, {}^2b/2(\exp({}^2b(2v-1)) + \exp(-{}^2b(2v-1))),\ v \in [0, 1]$$

where ${}^ia, {}^ib, {}^ic, {}^id$ are suitable constants determining form of curves.

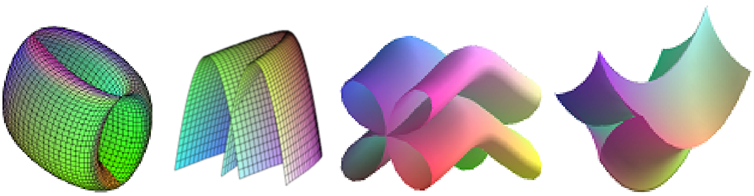


Fig. 2 Minkowski sums of pairs of various curves in $\mathbf{E}^3$

In Figure 3, axonometric views of Minkowski sum of an ellipse and an elliptical conical helix from the space $\mathbf{E}^4$ are illustrated with equation

$$\mathbf{r}(u,v) = \big(0,\, {}^1a(\cos 2\pi u,\, {}^1b \sin 2\pi u + {}^2a(1-v)\cos 2\pi v,\\ {}^2b(1-v)\sin 2\pi v,\, {}^2c 2\pi v\big),\quad (u,v) \in [0,1]^2$$

Resulting surface patch is a manifold in the four-dimensional space, which can be projected orthogonally into all four possible three-dimensional sub-spaces determined by triples of coordinate axes. These orthographic views are surface patches depicted by their axonometric views in all 4 different 3-dimensional spaces with particular triples of coordinate axes.

Minkowski linear combinations of two curve segments A and B are generated as Minkowski sums of k-multiple of curve A and l-multiple of curve B for any real numbers k and l, $C = k.A \oplus l.B$. Equation of linear combination of these two curves is

$$C : \mathbf{r}(u,v) = k\,{}^1\mathbf{r}(u) + l\,{}^2\mathbf{r}(v), (u,v) \in [\alpha,\alpha'] \times [\beta,\beta'],\ \ k,l \in \mathbf{R}$$

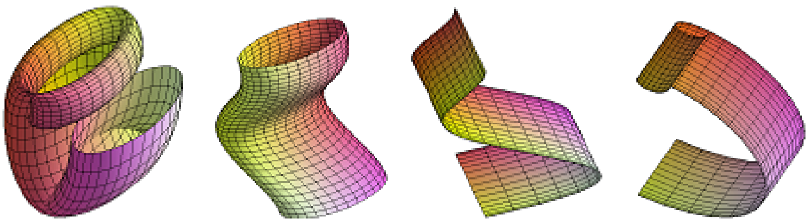


Fig. 3 Views of Minkowski sum of ellipse and conical elliptical helix in $\mathbf{E}^4$

Minkowski linear combinations of an ellipse and parabolic arc positioned in one plane and parameterised for $u = v$ with a choice of two different specific values of coefficients k and l are illustrated in Figure 4.

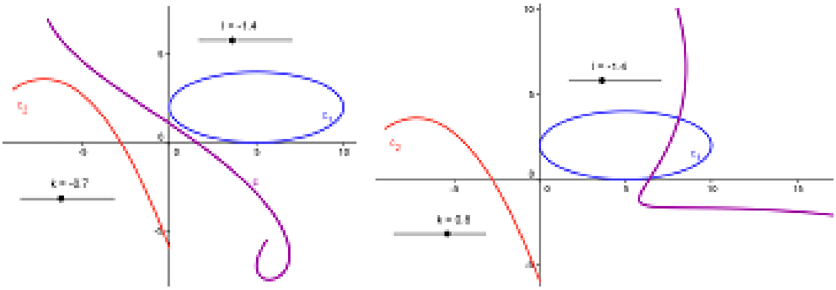


Fig. 4 Minkowski linear combinations of ellipse and parabolic segment

Surface patches illustrated in Figure 5 are Minkowski linear combinations of a lemniscate of Bernoulli

$$A:\ {}^1\mathbf{r}(u) = ({}^1a\cos u\sqrt{|\cos 2u|},\ {}^1a\sin u\sqrt{|\cos 2u|}, 0), u \in [0, 2\pi],\ {}^1a \in R$$

and parabolic arc

$$B:\ {}^2\mathbf{r}(v) = ({}^2av, 0, bv^2 + cv + d), v \in [\beta, \beta'],\ {}^2a, b, c, d \in R$$

These two planar curves are positioned in two perpendicular planes in the 3-dimensional space. Their Minkowski linear combinations are surface patches C generated as Minkowski sums of k-multiple of curve A and l-multiple of curve B for any real numbers k and l, with equation

$$\begin{aligned} C : \mathbf{r}(u,v) &= k\,{}^1\mathbf{r}(u) + l\,{}^2\mathbf{r}(v) \\ &= ({}^1ak\cos u\sqrt{|\cos 2u|} + {}^2alv,\ {}^1a\sin u\sqrt{|\cos 2u|}, l(bv^2 + cv + d)), \end{aligned}$$

for $(u,v) \in [0, 2\pi] \times [\beta, \beta']$.

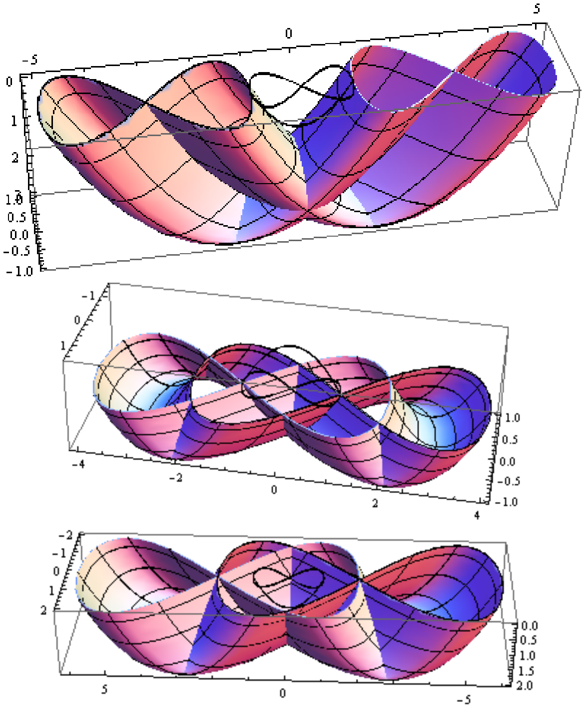


Fig. 5 Minkowski linear combinations of two curve segments in 3D

Minkowski linear combinations of two curve segments from higher dimensional spaces are well-defined and can model geometric figures in higher dimensions, details can be found in [3] and [4]. Orthographic projection from 4D to different 3D subspaces can be easily obtained. Some examples of Minkowski linear combinations of curves, i.e. various surface patches in 4D are presented in Figure 6 in their orthographic views.

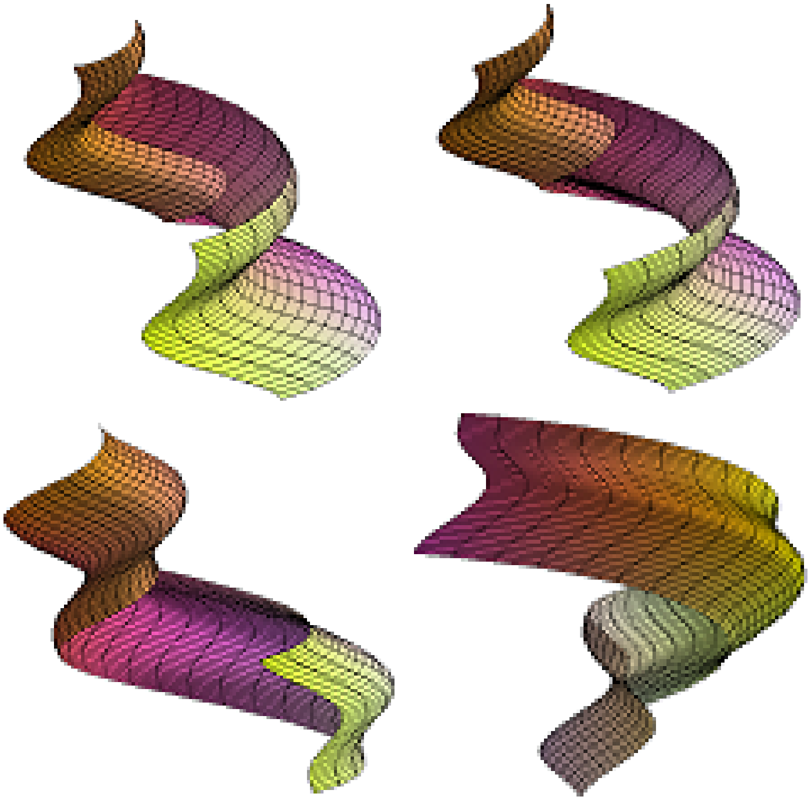


Fig. 6 Minkowski linear combinations of two curve segments in 3D

Examples of Minkowski matrix combinations of two plane curve segments with the same parameter are curve segments presented in Figure 7. Minkowski matrix combinations of asteroid and versiére in perpendicular planes are mapped in Figure 8, in the top row (their Minkowski sum is in Figure 2 on the right), while Minkowski matrix combinations of parabolic arc and lemniscate of Bernoulli are presented in the row below, for comparison with their Minkowski linear combinations in Figure 5.

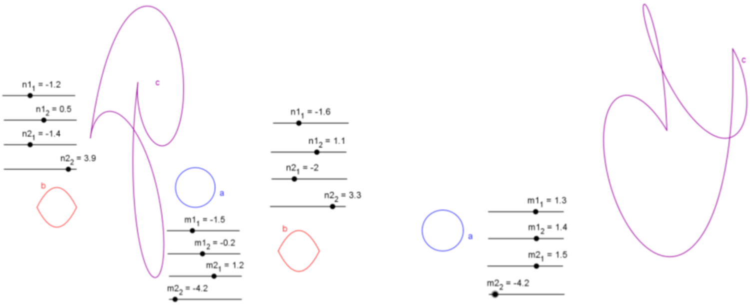


Fig. 7 Minkowski matrix combinations of plane curve segments

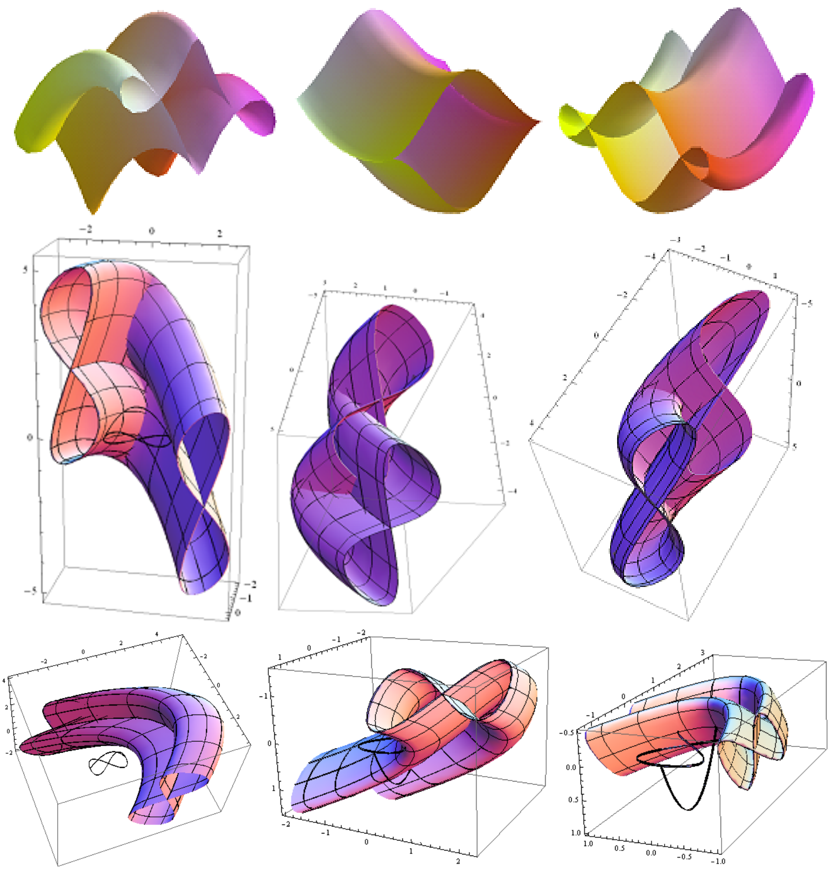


Fig. 8 Minkowski matrix combinations of curves in $\mathbf{E}^3$

2 Minkowski product of point sets

Minkowski product of two curves is the surface patch determined by vector equation that is the wedge product of vector equations of the two curves. Considering curve segments A and B defined by their vector maps

$$^1\mathbf{r}(u),\ \ u \in [\alpha,\alpha'], \qquad ^2\mathbf{r}(v),\ \ v \in [\beta,\beta'],$$

their Minkowski product is a surface patch C determined by equation

$$C : \mathbf{r}(u,v) = k\,{}^1\mathbf{r}(u) \wedge {}^2\mathbf{r}(v),\ \ (u,v) \in [\alpha,\alpha'] \times [\beta,\beta']$$

Example in Figure 9 shows the Minkowski product of a parabolic arc and the lemniscate of Bernoulli, in comparison to their Minkowski sum presented in Figure 5, at the top. Equation of this surface patch is

$$C : \mathbf{p}(u,v) = ({}^1a(bv^2+cv+d)\sin u\sqrt{|\cos 2u|},\ {}^1a(bv^2+cv+d)\cos u \sqrt{|\cos 2u|},\ {}^1a\,{}^2av\sin u\sqrt{|\cos 2u|}),\ \ (u,v) \in [0,2\pi] \times [\beta,\beta']$$

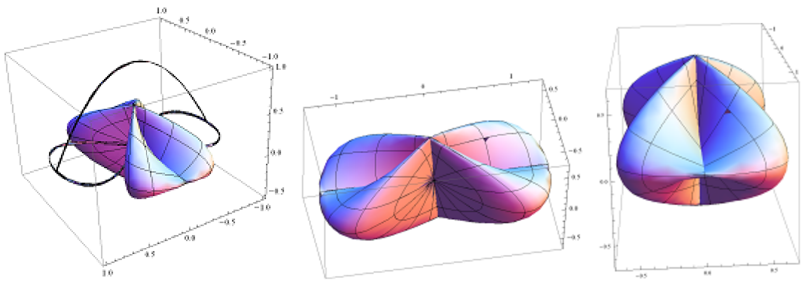


Fig. 9 Minkowski product of two curve segments in 3D

Minkowski product of differently parameterised shamrock curve and versiére that are located in different 3-dimensional sub-spaces of $\mathbf{E}^4$ shows a spectacular surface patch from the space $\mathbf{E}^4$ presented in different 3-D views in Figure 10. Equation of this surface has the following form

$$C : \mathbf{p}(u,v) = \big(a(2u-1)\cos v\sin^2 v,\ \frac{a\cos v\sin^2 v}{b+a(2u-1)^2}, 0, \frac{a\sin v\cos^2 v}{b+a(2u-1)^2}\big),\ \ (u,v) \in [0,1] \times [0,2\pi]$$

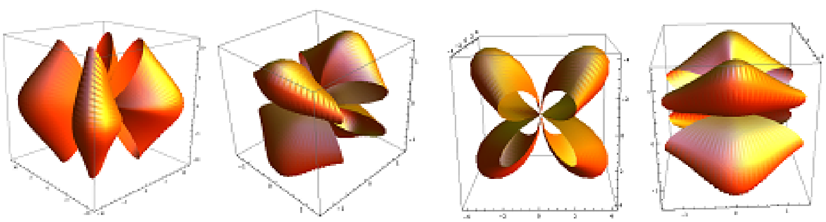


Fig. 10 Views of Minkowski product of two curve segments in 4D

3 Minkowski triples

Various Minkowski combinations of three point sets A, B, C can be introduced, in order to create unusual forms of point sets in $\mathbf{E}^n$ with specific properties determined by their generating principles or inherited from the original operands, smooth manifolds in 2 and even more dimensional spaces. Except for sums, combinations of sums and products depend on the order in which these operations are introduced.

Let us denote the Minkowski sum as $\oplus$, the Minkowski product as $\otimes$, and consider their vector maps:

A defined by ${}^1\mathbf{r}(u), u \in [\alpha, \alpha']$

B defined by ${}^2\mathbf{r}(v), v \in [\beta, \beta']$

C defined by ${}^3\mathbf{r}(w), w \in [\gamma, \gamma']$.

Various forms of surface patches can be obtained especially for equal parametrisations, in which either $u = v, u = w$ or $v = w$.

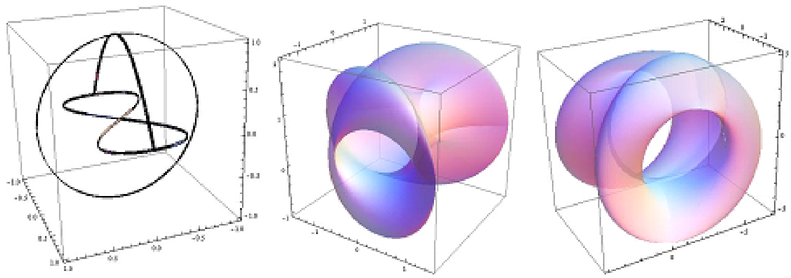


Fig. 11 Views of Minkowski sum triple of three curve segments in 3D

An example of surface generated as a Minkowski sum triple,

$$C = A \oplus B \oplus C : \mathbf{sr}(u, v, w) = {}^1\mathbf{r}(u) + {}^2\mathbf{r}(v) + {}^3\mathbf{r}(w),$$
$$(u, v, w) \in [\alpha, \alpha'] \times [\beta, \beta'] \times [\gamma, \gamma'],$$

of three curve segments positioned in perpendicular planes in $\mathbf{E}^3$, parabolic arc, circle and lemniscate of Bernoulli and given by equation

$$\mathbf{sr}(u, v) = (\cos u + \cos v \sin v, \cos v + \sin v, \sin u + \sin^2 v), \quad (u, v) \in [0, 2\pi]^2$$

is presented in different views in Figure 11.

Minkowski linear mixed combinations $(A \oplus B) \otimes C$ of three circles A, B, C located in perpendicular planes, while 2 of them are equally parameterised, are surface patches defined by

$$\mathbf{sm}(u, v, w) = ({}^1\mathbf{r}(u) + {}^2\mathbf{r}(v)) \wedge {}^3\mathbf{r}(w), \quad (u, v, w) \in [\alpha, \alpha'] \times [\beta, \beta'] \times [\gamma, \gamma']$$

and illustrated in Figure 14 and Figure 15.

Minkowski product triple $(A \otimes B) \otimes C$ is defined by

$$\mathbf{sp}(u,v,w) = ({}^1\mathbf{r}(u) \wedge {}^2\mathbf{r}(v)) \wedge^3 \mathbf{r}(w), (u,v,w) \in [\alpha,\alpha'] \times [\beta,\beta'] \times [\gamma,\gamma']$$

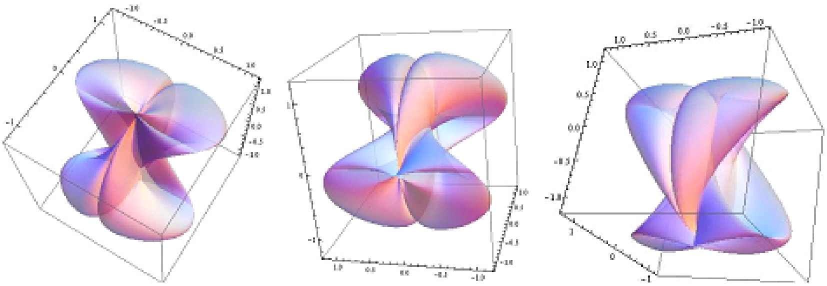


Fig. 12 Views of Minkowski product triple of three curve segments in 3D

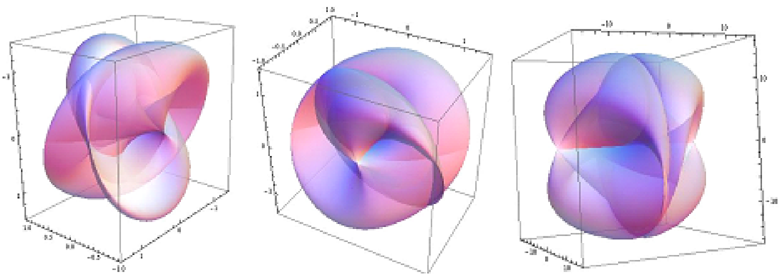


Fig. 13 Views of Minkowski mixed triples of three curve segments in 3D

Minkowski product triples of three curve segments with equation

$$\mathbf{sp}(u,v) = (-\cos u \sin v + \sin^2 u, -\sin^2 u(\cos u + \cos v), \\ \cos u \cos v + \sin v \cos^2 v), (u,v) \in [0,2\pi]^2$$

can be seen in Figure 12, while their Minkowski mixed triples with equation

$$\mathbf{sm}(u,v) = (\sin u(3\cos v + \sin v), \sin v(-\sin u \cos u + 3\cos u \sin v), \\ -\cos u(3\cos v + \sin v)), (u,v) \in [0,2\pi]^2$$

are illustrated in Figure 13.

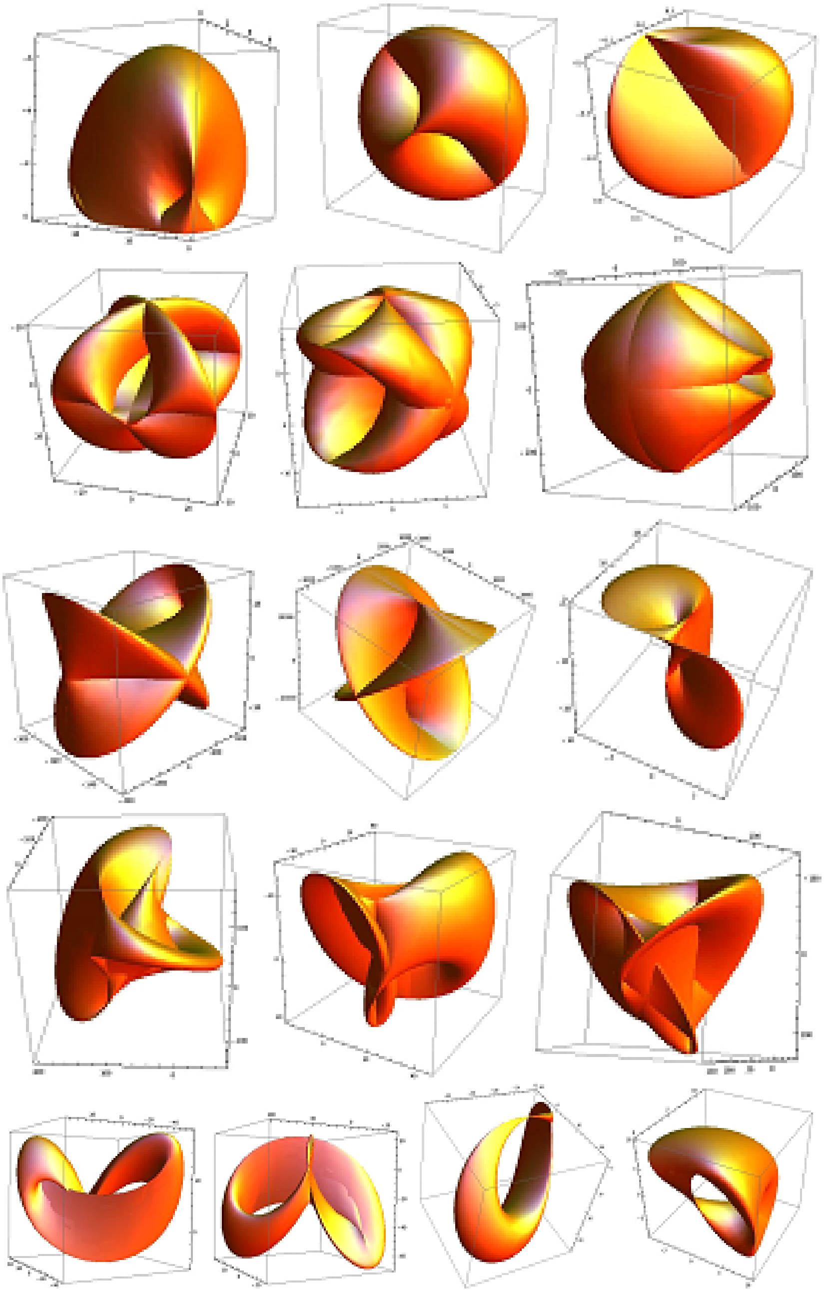


Fig. 14 Views of Minkowski product triples of three circles in 3D

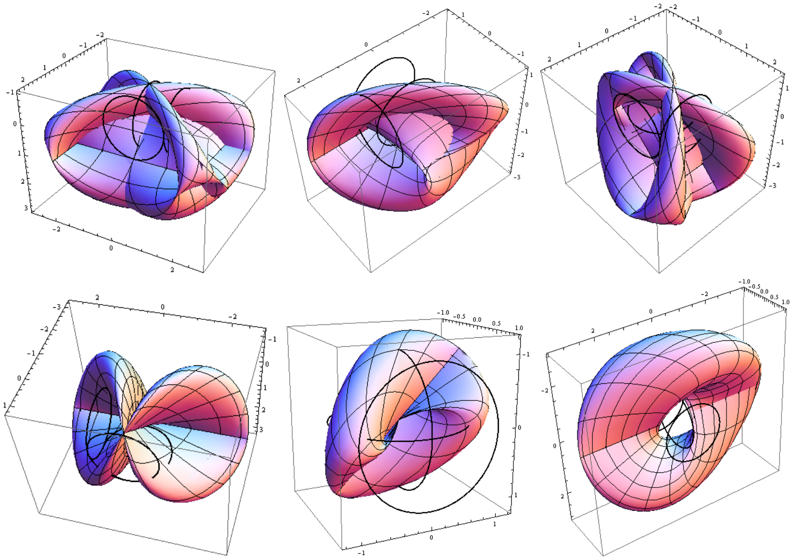


Fig. 15 Views of Minkowski mixed triples of 3 circles in various positions in 3D

Interesting shapes and forms of generated surfaces can be used for purposes of graphic design, in visualizations or in morphing. The underlying principles of Minkowski set operations provide a tool for the generation of both, synthetic visual and analytic representations. Thus the intrinsic geometric properties of the created objects can be studied using the methods of differential geometry and the specific properties inherited from the operand sets can be detected and defined. New objects are therefore interesting from both the aesthetic and the theoretical mathematical point of view.

Acknowledgment

The author would like to express her gratitude to Claude P. Bruter for his help in the presentation of the paper.

References

[1] M. Peternel, F. Manhart. "The Convolution of paraboloid and a Parametrized Surface", *Journal for Geometry and Graphics* 7, 2003, pp. 157-171.

[2] D. Velichová. "The Art of Geometry". In *Proceedings 2013, 16th Conference Bridges Enschede, Tesselations*, Phoenix, Arizona, USA, pp. 143-150.

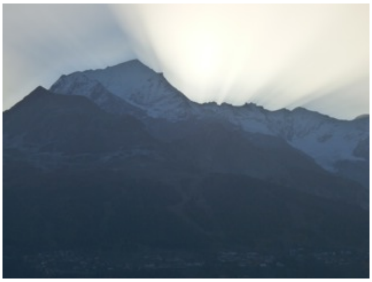
[3] D. Velichová. “Minkowski Combinations of Point Sets”, In *Proceedings of the 21st International Symposium on Computer Geometry*, SCG ´2012, SSGG, Kočovce 2012, SR, pp. 126-133.

[4] D. Velichová. “Minkowski Set Operators”, In *Abstracts, International Congress of Mathematicians,* Seoul 2014, pp. 152-153.

Professor of Applied Mathematics Head of the Institute of Mathematics and Physics
Slovak University of Technology
Nám. Slobody 17
812 31 Bratislava
Slovakia
daniela.velichova@stuba.sk

MATHEMATICS FOR THE WORKING ARTIST

Claude-Paul Bruter




Abstract

The meaning of the term “cone” defined in this article is much broader and more flexible than the classical one. Our extension of this concept lays the foundations for a broad mathematical theory that could be used by artists. This article is illustrated by examples taken from mathematical and botanical sources.The powerpoint [4] is a kind of summary of this article.

1 Introduction

“In a letter addressed to Émile Bernard dated 15 April 1904, Cézanne ambiguously writes: ‘Interpret nature in terms of the cylinder, the sphere, the cone; put everything in perspective, so that each side of an object, of a plane, recedes toward a central point.’ ” (From Wikipedia.)

Except during the Renaissance, painters have not studied and deepened the mathematics underlying their works. In the best cases, they have used what mathematicians have thought of and discovered.

In particular, many mathematicians have developed the study and the representation of their objects using numbers as a powerful coding system. Geometers and topologists use a more direct and intrinsic approach to define and understand these objects. Knots, polyhedra, spheres and tori have been the main fundamental objects they looked at and used to that aim.

In this article, I would like to focus the attention on the cones mentioned by Cézanne, and to what can be done with these cones. In the past, with the work by Apollonius and his successors involved in the theory of conics and quadrics, cones have played an important role in geometry, then, much later, in mechanics and physics. Mathematicians did not emphasize the fact that cones are also present in perspective theory, thus, in some sense, in projective geometry: remind the "central point" Cézanne was evoking.

The notion of cone I define and use here is much larger and flexible than the classical one. The introduction of different manners to assemble these cones through identification and attachment along singular elements allows the construction of a much richer collection of objects than the one obtained by the use of Cézanne's tools.

The article, illustrated by examples borrowed from the mathematical and the vegetal worlds, does not address the mathematician who would like to develop and expand the mathematical content along several directions [1] (projections, apparent contours, duality, transformations, enumeration, algebraic and numerical representations, sections, trajectories, in Euclidean spaces or not). It addresses the artist who might wish to play with all these cones and create new beautiful works for the pleasure of our eyes and of our mind.

2 Singularities

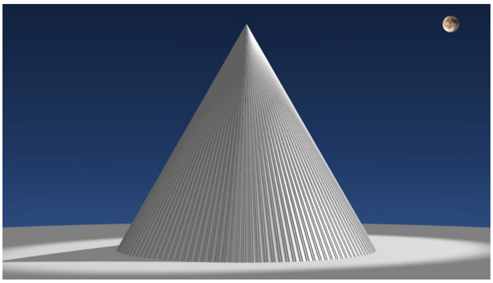
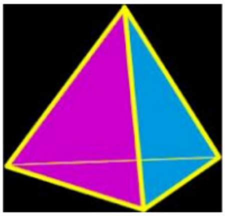


Fig. 1 A quasi standard cone, a view by Jos Leys.

[1]The mathematical theory behind this paper is the enormous theory of fiberspaces with singularities, whose first chapter is the theory of cones.

In a previous paper [1], several concepts and tools have been set forward, in particular the ones of singularity and of singular part of a shape.

Typical examples of singularities are for instance the vertices of a polygon in the plane, or the vertices of a polyhedron in the usual space, like the four vertices of tetrahedron, the six vertices of the octahedron (images from Wikipedia):



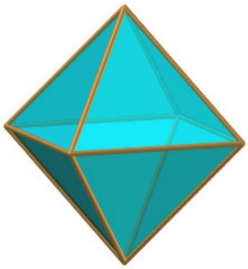


Fig. 2

D being a local connected domain of the shape, we shall say that it is homogeneous of dimension n, if any neighborhood of any point of D has the *topological dimension* n.

For instance:

- any edge of the tetrahedron, without the two vertices which close that edge, is a 1-dimensional domain;
- any face of the tetrahedron, without the triangle which borders it, is a 2 dimensional homogeneous domain.

Any subdomain of D whose topological dimension is $k < n$ is a *potential singular part* of D.

Thus, any point ($k = 0$) of a face of a polyhedron is a potential singular point, any curve ($k = 1$) drawn on the face, is a potential singular part. Vertices of polygons and of polyhedra are not only potential singularities. They will be defined as *(incarnated) singular points*.

3 Cones

3.1 Introduction Traditionally, there is nothing inside a tetrahedron: it is an *hollow object*. But it may be filled with matter: it becomes then an heavy die, a *full object*. We shall make the distinction between hollow cones and full cones.

In this paper we shall consider Euclidean spaces only. All the cones **C** we are going to consider will be here defined[2] by the three following ingredients, the two first ones a priori lying in an n-dimensional Euclidean space:

1) a *vertex* V (topological dimension 0), called the *main vertex* or the *apex* of the cone.

2) a *basis* denoted by $B(f)$, a closed domain of topological dimension $n-1$, or by $B(h)$, which is the boundary of $B(f)$, thus a closed domain of topological dimension $n-2$. [In the following example (Figure 3 left), $B(f)$ is supposed to be a full triangle].

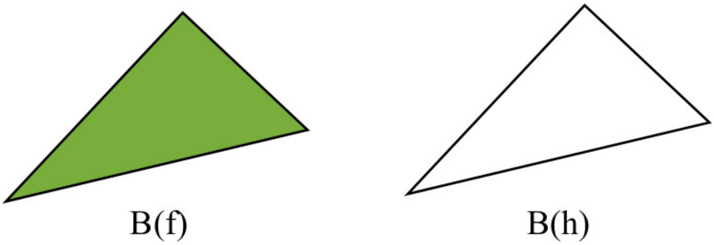


Fig. 3

3) an interval $I \subset \mathbf{R}$ (topological dimension 1) called the *standard generator* or *fiber*, each of whose inclusions into the cone through a non decreasing differential mapping: $i_P : I \to \mathbf{R}^n$ is a curve that joins a point P of the basis to the apex V. This curve is called the *local fiber* at P.

Definitions: A cone $\mathbf{C}$ with $B(f)$ as a basis is called a *full cone of dimension* n. A cone with $B(h)$ as a basis is called an *hollow cone of dimension* $n-1$. We shall say that a *cone is linear* if all the local fibers are intervals (all the local inclusions are linear mappings).

3.2 Standard basic examples

3.2.1 As an example of full cone, we may choose the full tetrahedron. We can look at it as a (linear) cone if we:

– choose a vertex of this tetrahedron and name it V.
– consider the opposite closed face to V - its topological dimension is 2 - and name it $B(f)$.
– consider the intersection of the tetrahedron with any line which cuts $B(f)$ at any P and joins V. This intersection is the local inclusion of the interval I, the fiber at P.

[2] This definition can be understood as the result of the Bourbakist point of view: Bourbakist in the good sense, i.e. a structuralist point of view, looking at the elements of an object which characterize its structure.

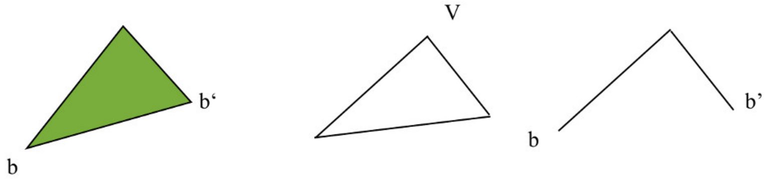


Fig. 4

The topological dimension of the full tetrahedron is 3 as being equivalent to a full sphere called a 3-dimensional ball. The boundary of this cone is the *complete* hollow cone associated with the full cone.

3.2.2 Now, from the full tetrahedron, we can extract an other hollow cone by considering the three faces adjacent to V:

- V remains the apex of that cone,
- its basis $B(h)$ is now the close curve, i.e. the hollow triangle which bounds $B(f)$, the opposite face to V,
- the intersection of the tetrahedron with any line which cuts $B(h)$ at any P and joins V. This intersection is the local inclusion of I.

3.2.3 Simpler, Figure 4 shows a green triangle which is a full 2-dimensional linear cone lying in the usual plane. Its basis bb' is the opposite side to the apex V. The boundary of bb' is the set of the two points b and b'. The two hollow 1-dimensional corresponding cones appear on the right.

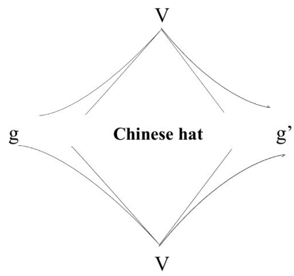


Fig. 5

3.2.4 When $n = 2$ (the plane), Figure 5 shows the example of four hollow 1-dimensional cones whose vertex V is an antibubbling singularity (up) or a bubbling singularity (down). The basis here has two points which are not visualized here. The curves g and g' are local inclusions of the interval I. I shall call that cone a "Chinese hat". In each case, one sees two cones with the same apex V: the larger one is non linear, the linear one comes out from the previous one by considering the tangent lines in V to g and g' respectively.

3.2.5 Consider any family of knots in the usual 3-space (simpler, a pencil of conics), points in that space which play the rôle of apices: look at the mountains you obtain!

4 A few remarks and definitions

4.1 Any n-dimensional full cone $\mathbf{C}$ with apex V and basis $B(f)$ gives birth to two $(n-1)$-dimensional hollow cones with apex V: the complete hollow cone of $\mathbf{C}$, which is the boundary of $\mathbf{C}$, and $\mathbf{C}^c$ the *coat* of the full n-cone whose basis is the boundary of $B(f)$. This coat is included in the boundary of the full cone.

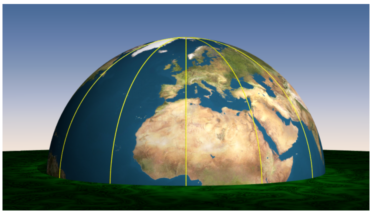
Conversely, a hollow $(n-1)$-cone with basis $B(h)$ can be the coat of an infinity of n-cones. Any two such n-cones share the same boundary $B(h)$ of their respective basis $B(f)$ and $B(f)'$. They wear the same coat. These n-cones will be named the *wearers* of the $(n-1)$-cone.

4.2 Figure 5 shows the example of a linear cone which is defined by the tangents at its vertex V to the fibers of a given cone, with the property that the angle between the tangents is not null, nor equal to π.

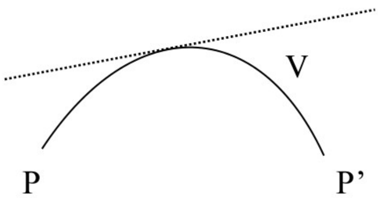
Cones with such a property, i.e. the tangent cone is not a linear $(n-1)$ subspace, will be called *rough cones*.

A rough cone has a *unique linear tangent cone*.

But conversely, a linear cone has an infinity of rough cones for which it is their common linear cone.



Half-sphere as a soft 2-cone in the 3D-space:
its basis is a circle.
A view by Jos Leys.



Arc of circle as a soft cone:
an edge joining its basis
$B(h) = \{P, P'\}$.

Fig. 6

4.3 If the tangent cone is such a $(n-1)$ linear subspace, the cone is a *soft* or *spherical cone*. Any point of an half-circle, of an half-sphere, is thus a *spherical apex* of a soft cone and a potential singularity. It becomes an incarnate singularity when its location becomes defined by the supplementary data of a directional line for instance.

A particular interesting situation happens when the main vertex of a cone is located on its basis. In that case, we shall speak of a *basic cone*. Basic 1-cones play a fundamental role.

4.4 Let us consider the 1-dimensional hollow cone named the *cusp* [3] and defined by:

- the apex $V(0,0)$ is the origin of a usual orthogonal coordinate system of the real plane,
- the basis $B(h)$ of this cone is the set of the two points $P(1,1)$ and $P'(-1,1)$,
- I is the interval $[0,1]$, and the local inclusions of I in P and P' respectively are defined by the parametric equations:

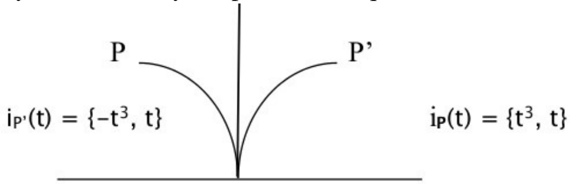


Fig. 7

The line which joins the apex V to any point $Q(x,y)$ of the fiber $i_P(I)$ has a slope defined by the ratio $y/x = t/t^3 = 1/t^2$. When t tends towards 0, this slope becomes infinite, so that the tangent in V to this fiber is the vertical line.

For a similar reason, the tangent in V to the fiber at P' is also the vertical line.

In other words, that cone has a unique vertical tangent in V: the linear tangent cone is an half linear space.

A cone whose linear tangent cone is so degenerated will be called a *penetrating cone* or a *spine*.

4.5 Let us go back to the examples illustrated by Figures 5 and 7. In Figure 5, the upper cones seems to be the symmetric of the under cones with respect of the horizontal line. More generally, any cone has a *symmetric* one with respect to any domain parallel to the domain containing its basis.

[3] The cusp is the most basic singularity. It has been used as a geometrical support in a study of the universal phenomenon of ambiguity.

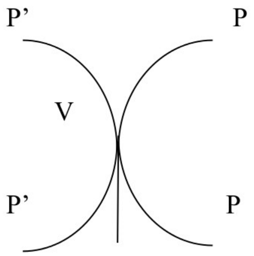


Fig. 8

4.6 Let **C** be a given full n-dimensional cone with vertex V. Let B be a n-dimensional ball whose center is V: the boundary of that ball is the $(n-1)$-sphere centered in V. We suppose that the ball is small enough so that the common part to the ball and the cone is entirely contained in the cone.

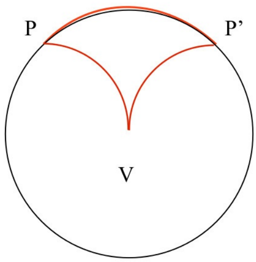


Fig. 9

The *complement* **C*** of **C** in B is a full n-cone with the same coat as **C**.

Let us consider the "half" n-spaces through V. If **C** is contained in one such half-space, **C** will be named the *male part* of B, and called a *male cone*.

Its complement is the *female part* of B and is a *female cone*.

4.7 Let $\mathcal{F}$ be a given continuous family of $(n-1)$-cones $\mathbf{C}_t$ parametrized by t belonging to I, with apex V_t and with the same basis B. Let $\mathcal{A}$ be the curve $t \to \mathcal{A}(t) = V_t$: this curve will be called *an axis* of the family.

Fig. 10

Let T be a $(n-1)$-dimensional domain which is transverse to the axis, and σ_t be the section of the cone $\mathbf{C}_t$ by T. We suppose that for any $t' < t$ the closure of σ_t' is contained in the closure of σ_t.

Then the closure of $\mathcal{F}$ is a n-dimensional cone *foliated* by the cones $\mathbf{C}_t$.

Any $(n-1)$-cone $\mathbf{C}$ is the coat of an associated canonical wearer $\mathcal{F}(\mathbf{C})$. An axis of $\mathcal{F}(\mathbf{C})$ be also called an *axis* of $\mathbf{C}$.

Discrete foliations of cones can be worked out in the same spirit.

4.8 Let $\mathbf{C}$ be a 1-dimensional cone, P and P' the two distinct points of its basis. Let us suppose that the curvatures at any point of the fibers are not null or infinite except maybe in V.

Such a cone, like the left one, might be named a *half smiling cone* if these curvatures have the same sign.

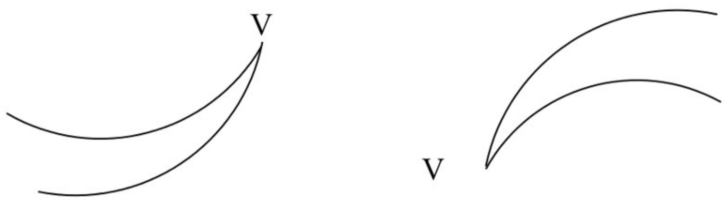


Fig. 11

4.9 Let $\mathbf{C}_i$ be an $(n-1)$-dimensional cone embedded in a n-space and called the *motive*, $h(\mathbf{R}) = \Lambda$ be a curve of such a space (more generally a $k < (n-2)$ dimensional domain), and V a point of Λ. Let S_i the shape defined by $S_i = \Lambda \times \mathbf{C}_i$ so that V is the apex of a cone $\mathbf{C}_i$.

The shape S_i will be called a *regular conical excrescence* of $\mathbf{C}_i$ along Λ, Λ being its *singular curve* or again its *handle*. Note that several S_i can share the same singular line, so that the union $S = \bigcup S_i$ of these local shapes can be taken into consideration.

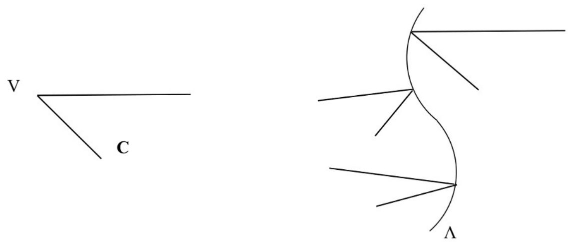


Fig. 12

(More generally, we may suppose that, for each V, the corresponding cone is subjected to an eventually continuous controlled change of metrical properties).

Given a curve Λ in a n-dimensional space, a point V of that curve for which the tangent to the curve is well defined, a transversal subspace to the curve in V is a $(n-1)$-dimensional subspace which does not contain the tangent.

Conversely, suppose that V belongs to a shape S so that any transversal subpsace to V defines a cone $\mathbf{C}_v$ on S whose main vertex V is on Λ, then Λ is defined as a *singular curve* of S. When Λ lies on a cone, S will be named a *flag*.

When the cones are full cones, we shall say that S is a *mountain* and Λ its *line of summits*.

A fairly nice mathematical example of the coat of such a mountain is the Whitney umbrella where Λ is a line:

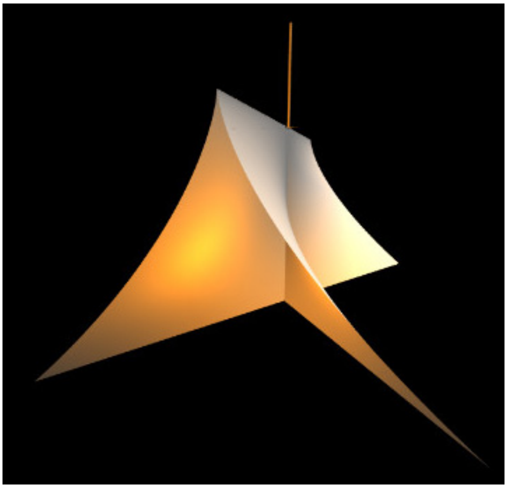


Fig. 13 The Whitney umbrella (from www.algebraicsurface.net)

4.10 Let $\Gamma(\Delta)$ the group of symmetries of a part Δ of the basis of a cone $\mathbf{C}$. Δ induces the part $\mathbf{C}/\Delta$ of the cone, and $\Gamma(\Delta)$ will be called the *symmetry group* of that part $\mathbf{C}/\Delta$.

4.11 Indeed, the way according to which cones are attached to singular domains is not restricted to the consideration of their main vertices. Any other singular part of dimension $k' < k$ of a $(n-1)$-cone, where $k < n-2$ is the dimension of a domain Λ, can be attached to Λ.

Fig. 14 The “leaf” of a fen is called a penna, which is made of pinnulae, here viewed as 2-cones, whose basis are attached to the singular curve named the rachis

5 Compositions of cones

5.1 1-dimensional cones

5.1.1 Introduction Let us first give a list of non spherical 1-dimensional hollow rough cones in a flat 2-dimensional space. Each one has tow edges: one of them will be called the *arm*, the other one the *anti-arm*.

Fig. 15

(It is amazing to compare this list published in 1976 [2], with the following by Dürer around 1528 [3]: $1976-1528=446$.)

Fig. 15 bis Dürer’s list

Let us add to that list a soft 1-dimensional hollow cone, like an half circle or an edge, a spine like the cusp, and the basic 1-cone, an edge whose one vertex is the main vertex of the cone.

Each cone of the list gives rises to a series of n-folded arms cones like this elementary one:

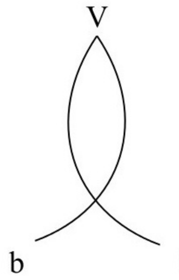


Fig. 16 1-folded arms

The basis of each of the cones $\mathbf{C}_i$ of the original list is a set of two points: $\{P_{i1}\ ,\ P'_{i1}\}$. Each such cone gives birth to an infinity of wearers (full cones) which can also be taken into consideration.

The boundary of a full 2-cone has: three points, the apex V and the two vertices of its basis, the two curves of the hollow cone that join the vertices of the basis to V (the arm and the anti-arm), and the curve of its basis $B(f)$ which joins the two vertices of its basis. All these curves may be viewed as singular elements of the full cone.

More generally, a fiber Σ of the cone is *singular* if it contains an element of curve Λ such that the intersection of its neighborhood with the cone is a flag S whose local conic motive is rough or a spine.

We are going to proceed to attachments of 1-cones along these different singular elements through processes of identification.

5.1.2 Self-attachment Given a 1-cone, the identification of the two points of its basis gives rise to a topological 1-sphere like a circle, while the identification of the two edges adjacent to the main vertex gives rise to a basic 1-cone.

Fig. 17

5.1.3 Attachment of cones by identification of their apex The attachment by identification of the apex of two basic 1-cones gives birth to one of the previous 1-cones.

Cones attached to each other by identification of all their apex to one of them will be called *spiders*. In that case, each cone of the spider could be called an *arm* or a *tentacle*. Figures 8 and 18 show examples of particular spiders.

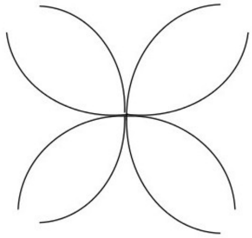


Fig. 18 Spider or Flower or Bouquet

5.1.4 Attachment by identification of a unique point of their basis

5.1.4.1 Let Σ be a sequence of N various cones of the list, any cone $\mathbf{C}_i$ appearing n_i times in the sequence. Then two consecutive cones $\mathbf{C}_i$ and $\mathbf{C}_j$ - where j can be equal to i - are attached by a unique point of their basis, if only one point of the basis of $\mathbf{C}_i$ is identified with one point of the basis of $\mathbf{C}_j$.

In that way, we shall say that we have got a *garland* or *frieze* of 1-cones, or a flag if all the cones except one of them called the *handle* are attached to this handle.

If the first cone of the sequence is attached to the last cone of that sequence, we shall say that the garland is *knotted* or *polygonal*: we can understand a knotted garland as as a knot with singularities.

Fig. 19 $\mathbf{C}_1$ is attached to $\mathbf{C}_2$ which is attached to $\mathbf{C}_3$ which is attached to $\mathbf{C}_1$

A polygonal garland with $2N$ edges can be constructed with N cones.

Polygonal curves with an odd number of edges $2n+1$ may need n rough or penetrating cones plus a soft cone. But an other way to get such a polygonal garland is to divided each edge into two attached parts, and then to use $2n+1$ cones to get it.

Each knotted garland generates a spider, its dual, but the converse is not always true.

Except lines, any other curve in any n-dimensional space can be decomposed in such rough or penetrating hollow 1-cones, so is a garland of hollow 1-cones.

5.1.4.2 Here is an example arising from the mathematical butterfly in catastrophe theory. The following local section of this surface can be viewed as a garland of the two following cones:

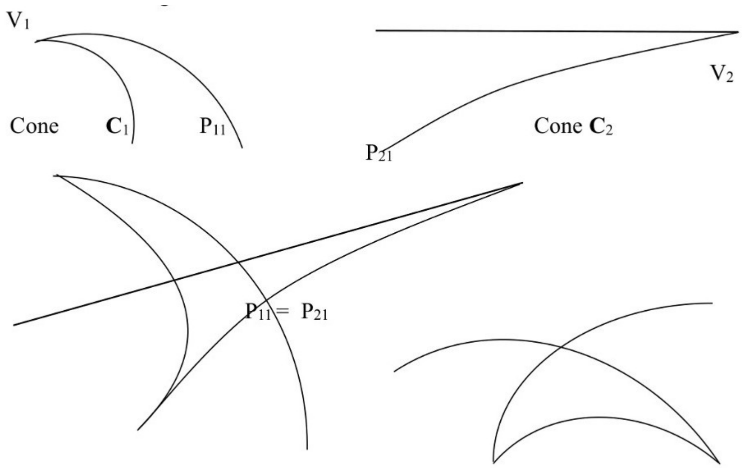


Fig. 20 The Bird, the Swallow Tail

Appropriate deformations of the above drawing give birth to a stylization of a bird.

The following shows a stylized fish as, first, the visualization of a white 1-dimensional full cone where all the fibers have a unique other common point than the vertex - but of course they could have many such common points.

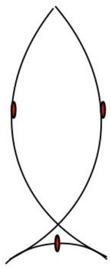


Fig. 21 The fish

But if we introduce fictive or virtual vertices in the middle of each edge (the red points on the figure), we then define three hollow 1-cones with main vertex respectively V, b and b' from which the fish can be reconstructed.

5.1.4.3 Suppose a given 1-cone imbedded in a n-dimensional space. The possibilities to attach an other 1-cone to one point of the basis of the given cone is infinite, being ruled by the group of rotation of that n-space. Given constraints can of course reduce this set of potential possibilities.

5.1.5 Attachment by identification of the two singular points of their basis Base of 1-cones are very elementary. Given a process of attachment (the choice of the manner to identify the basis), there are infinite possibilities of attachment of cones to a given one imbedded in an n-space, each possibility being defined here by an element of the group of rotation of the $(n-1)$-space.

We shall call a *p-flag* the set of $(p-1)$ 1-cones so attached to a given 1-cone, the handle.

Here is an easy example in the plane ($n = 2$).

The given 1-cone is $\mathbf{C}_1$, while the 1-cone to be attached to it is $\mathbf{C}_2$, indeed a clone to the first one:

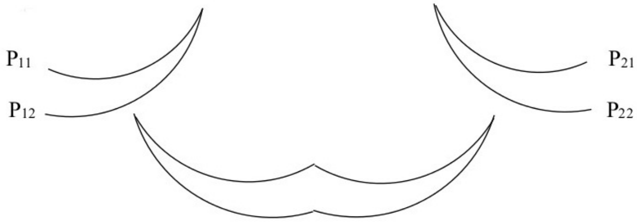


Fig. 22 Smile and Moustache

There are only two ways to attach the two cones with the same identification of their basis. The first one gives a perfect superposition of the two cones since they have the same shape (identity of $O(1)$). The second way, a symmetry, gives rise to a true smile; or a moustache!

Here is an other way to construct the fish where a point of the basis of a first cone is the apex of a second cone.

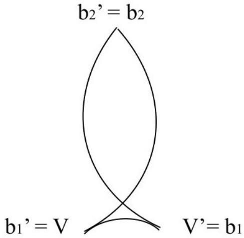


Fig. 23 The contour of a fish built from two symmetric 1-cones (can also be the complete hollow 1-cone of a fish).

5.1.6 Attachment along singular curves We have been considering attachments along the apex V and the elements of the basis. We now consider identification of the singular curves joining V to the vertices of the basis, two such curves being able to be identified if and only their curvature is the same.

Given two red 1-cones with apices V_1 and V_2, this identification first implies that the identification of V_1 with V_2, and the identification of a vertex b_1 of the basis B_1 with a vertex b_2 of the basis B_2: in other words, the attachment 4.1.2 and 4.1.3 have be done simultaneously, but that is here a part of the process since the identification concerns all the points of the singular curves.

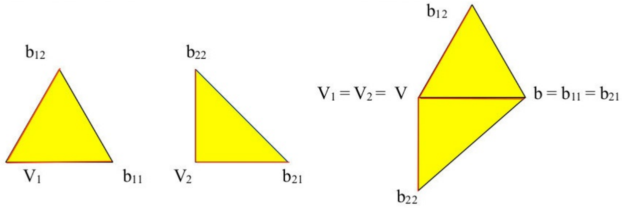


Fig. 24

A sequence of N 1-cones in an n-space ($n > 2$) which are attached along a singular curve g of a given one will be called a *N flag along g*.

The use of less usual 1-cones gives birth to unusual shapes, especially if all the processes of attachement are used all together.

5.2 2-dimensional cones

5.2.1 Introduction and examples

5.2.1.1 First, let us show a very few 2-cones - the mathematical images are borrowed from the net, see for instance “images of algebraic surfaces”:

The same operations of identification and attachment can be worked with n-dimensional cones. Here are a few classical pictures of assemblies of 2-dimensional cones attached along singular parts of their boundary, apices, edges, basis:

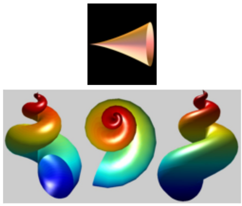
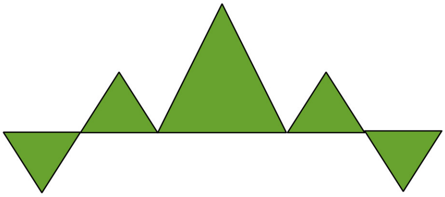
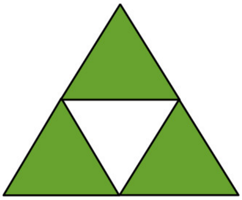


Fig. 25 (Images from the web)



A garland (or frieze)



Serpinski motive as knotted garland

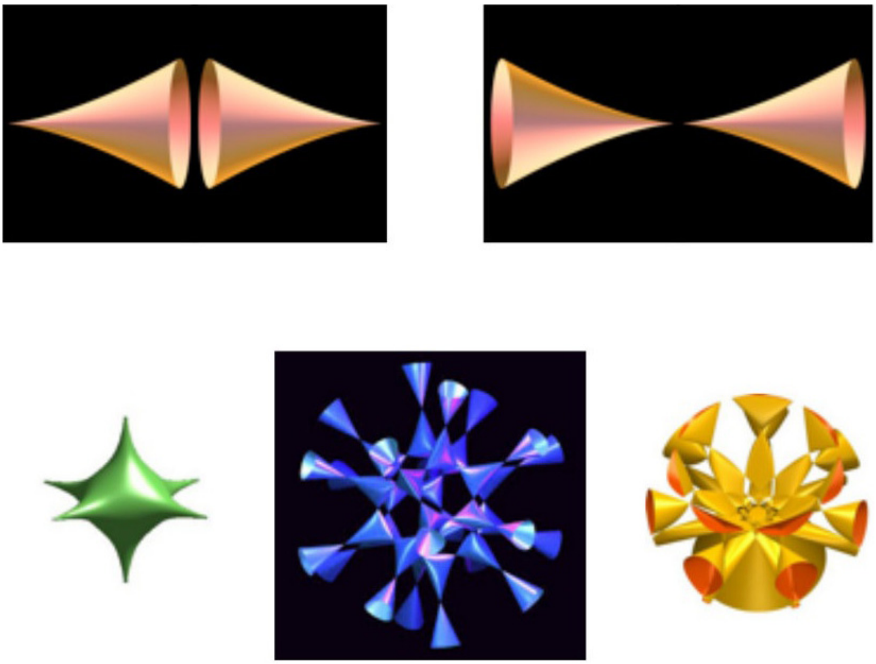


Fig. 26 Classical nodal surfaces

It is easy to extend these examples by adding more cones of different sizes, or to start with other polyhedra including Gosset polyedra, using apices defined through discrete subgroups of $O(n)$, and reproduce similar constructions of attached cones.

5.2.1.2 The vegetal world is also a source of examples. Let us first consider the following standard mathematical 2-cone and one of its incarnation as a leaf of the lily of the valley in the usual 3-space:

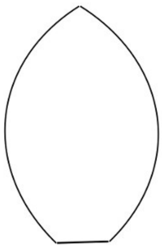


Fig. 27

This incarnation has the nice property to show possible fibers of the cone. Nature is now going to attach along their basis two clones of that cone. Here they are:

Fig. 28

5.2.1.3 Let now us consider the following mathematical smiling 2-cones and the two following leaves:

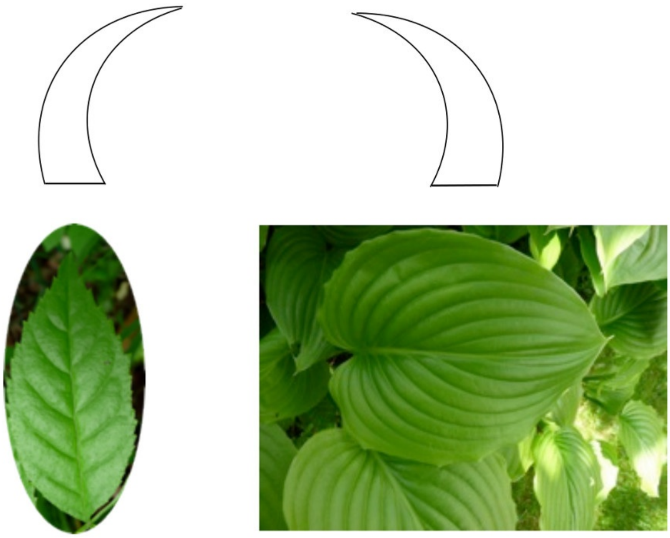


Fig. 29

The left leaf shows two similar sequences of half smiling 2-cones, more or less symmetrically located on a singular curve like in Figure 14. But each central cone is attached along a singular line of its border to two another cones, one above and the other under itself. On the leaf of the right, moreover, all the apices meet at the top of the leaf, on the singular curve. Indeed, these cones getting very thin give rise to the “fibers” which appear on Figures 22 and 23.

5.2.1.4 Let us know consider the following leaf:

Fig. 30

We discover that the so-called previous generic 2-cones which seemed to appear on Figure 30 say are indeed mountains in the sense we used to caracterize the Whitney umbrella (Figure 13).

5.2.1.5 Let us give here a few simple other examples of 2-dimensional objects created with more simple 2-cones using the standard techniques of attachment:

For instance, the 2-cones of Figure 25 can be created by the classical identification of the two edges adjacent to the apex of convenient "triangular" 2-cones: cutting and opening the given 2-cones along a curve through their apex give rise to the convenient triangular 2-cones.

The standard 2-band in the usual space can be created by attachment of two triangular 2-cones **C** and **C'** like full half-smiles, but which can have any specific shape:

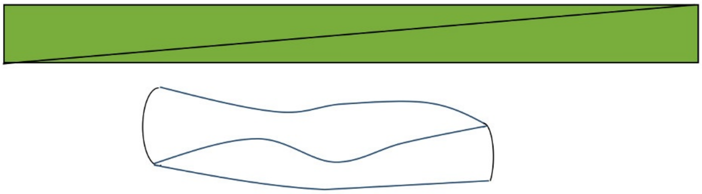


Fig. 31

Twist the band as you wish in the usual 3-space, attach the corresponding opposite sides and get Möbius bands, deformed cylinders and tori.

5.2.1.6 There are many ways to assemble cones of different shapes and to create landscapes.

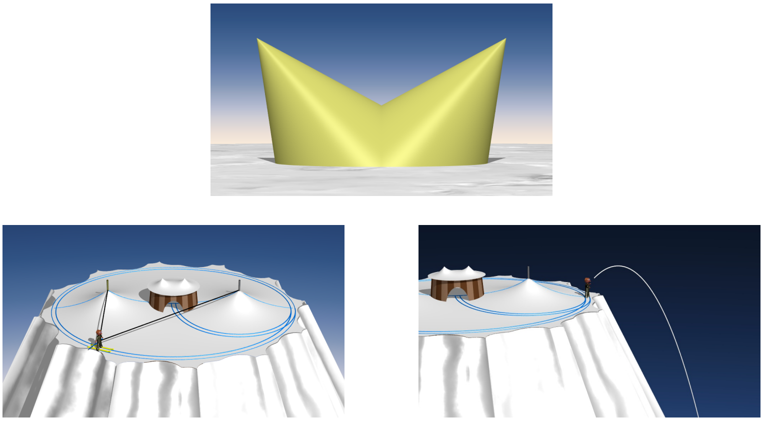
Here are two examples of such constructions: the first one, among the simplest, show a double cone arising from the identification of the basis of two linear cones in the usual space, the second one was made by nature, a few years ago.

5.2.1.7 Here is a final remark about 2-cones, one that more generally applies to n-cones. In the usual 3-dimensional space, let **C** be a hollow 2-cone with basis $B(h)$, D a 2-dimensional linear subspace which meets that cone. The common part of **C** and D is a plane curve β. This curve may have singularities and multiple common points. Then the part of the fibers through these points between D and the apex V of the cone are singular curves of the cone.

5.2.2 Creation of 2-cones from 1-cones We are now going to look at two main techniques to create 2-cones from 1-cones.

5.2.2.1 From full 1-cones, by attachment:

Let us first recall that a full 1-cone is indeed a 2-cone since it is a 2-dimensional surface.



Double cones by Jos Leys

Fig. 32

The hollow tetrahedron gives an example of the attachment along singular lines of a sequence of 3 standard linear full 1-cones:

$$(V_1, b_{11}, b_{12}), \qquad (V_2, b_{21}, b_{22}), \qquad (V_3, b_{31}, b_{32})$$

attached one to the other through the identifications of the singular lines

$$(V_1, b_{12})\texttt{@}(V_2, b_{21}) \qquad (V_2, b_{22})\texttt{@}(V_3, b_{31}) \qquad (V_3, b_{32})\texttt{@}(V_1, b_{11})$$

More generally, we shall call a polyhedral 2-cone such a 2-cone constructed from a sequence of full 1-cones with a cyclic presentation of their singular lines. Note that generically, this kind of 2-cone is not a standard polyhedron nor a part of such a polyhedron.

Flags of 2-cones can be constructed by attaching other 2-cones along a singular line of one of them, or along the basis of one of them, or along a curve of excrescence.

5.2.2.2 From hollow 1-cones, by local transformations:

1) Let **C** be a $(n-2)$-cone in an n-dimensional space, A be a curve which contains the apex V. We denote by A_C the set of points Q of A for which L_Q the linear orthogonal $(n-1)$-dimensional subspace to A in Q meets **C**. Denote by $\mathbf{C}(L_Q)$ the intersection of **C** and L_Q.

Let ρQ be a continuous translation and/or a continuous rotation of $\mathbf{C}(L_Q)$ around Q in the subpsace L_Q giving birth to the trace $T\mathbf{C}(L_Q)$ of $\mathbf{C}(L_Q)$ in that subspace. We suppose that ρQ is a continuous function of Q. From the geometric point of view, we can also suppose that each local transformation changes the local sizes.

The union of these traces $T\mathbf{C}(L_Q)$ when Q moves continuously on A_C is a $(n-1)$-cone.

Here is a trivial example where ρQ is a 360° rotation, A is a vertical line. Starting with an half smiling cone, we may get for instance the following hollow 2-cone. We might call the corresponding full 2-cone the “bell”, or the “hat”.

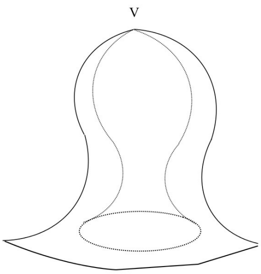


Fig. 33 The bell

2) More generally, A does not contain V. Then we do not get a cone in general, but the coat of a moutain.

A fairly simple example is the Whitney umbrella that can be obtained by translating a Chinese hat, without any metrical transformation of its size.

From the metrical (geometrical) point of view, the presence of local symmetries of the basis is of some interest. One can impose in particular that the vectorfield which acts on the transversal sections $\mathbf{C}(LQ)$ keeps on the associated group of symmetries. Then we get a privileged axis.

5.2.3 Full 2-cones, the 3-ball and the 2-sphere Let us consider a family of full linear 2-cones **C**(t) like full triangles. From the topological point of view, one can represent them by 2-cones whose basis are arcs of circles.

Let $A(t)$ the area of the cone **C**(t). We suppose that the mapping $t \to A(t)$ where t describes the interval $[0,1]$ is continuous, with $A(0) = 0$, and $A(1) = A$.

Now, in the usual 3-space, let Λ a vertical interval, and S the shape, the flag defined by a regular conical excrescence of the family of **C**(t) along the handle Λ.

Here is (left) a vegetal example of such a shape showing **C**(1) and Λ, together with its symmetric (right):

Fig. 34 Flags with homothetic cones

Let $1(t)$ and $2(t)$ be the arms of **C**(t) and call the sets $F(i) = \{\bigcup\{i(t) \mid t \in [0,1]\}$ the i-face of S, where $i = 1$ and 2.

Consider n clones of S, S_1, S_2, ..., S_n, and their respective faces $F_k(i)$ where $F_k(i)$ is the i-face of the clone S_k.

Identify their handle to get a flag, then identify the face $F_k(2)$ with the face $F_{k+1}(1)$ for $k < n-1$, the face $F_n(2)$ with the face $F_1(1)$.

Topologically, the result is a 3-ball whose boundary is the 2-sphere: one may taste an equivalent final following conclusion.

Fig. 35

6 Singularities again

6.1 Creation The pinching process [1] is a standard process to create singular sub-domains. The creation of a singular point can be practically worked out in the following way. Choose the location in the object close to which the singular point should appear. Consider a small ball containing this location and a point V inside the ball but out of the object. The intersection of the ball with the object will be the basis of a hollow cone with apex V such that the object and the cone share the same tangent space along the basis. Attach the cone to the object and cut off the interior of the basis.

Note that when the object is locally convex, the resulting singularity V can be bubbling or anti-bubbling according to its position with respect to the object.

A physical equivalent way to create a singularity consists in choosing a point V on the object and to draw out the object along curve through V. Such a process has been for instance used by Philippe Charbonneau to create the following sculpture:

Fig. 36 Biconique 2 by Philippe Charbonneau

Here, the object is a curve, the knot called the trefoil which bounds a Möbius band. The curve was drawn out at two points V and V' which have been fixed up on a vertical rigid axis.

More complex sculptures could be similarly worked out with any other regular torus knot.

6.2 Suppression

6.2.1 The first natural process is to smooth the object by suppressing locally the cone and substituing to it a small half ball or half sphere. We may call this process the rounding of the singularity.

I shall show a very few reasonably good home made photos first for the pleasure of the eyes.

Fig. 37

The first group of photos illustrate the internal symmetry of flowers and the layout of their petals viewed as cones. Indeed, it seems to me that the main symmetries of the floral world are of order: 2, $2+2$, 4, 3, $3+2$, 5. $2+2$ means a superposition of orthogonal symmetries of order 2. Similarly, $2+3$ means a superposition of a symmetry of order 2 and a symmetry of order 3. Frequently, the order of these fundamental symmetries is multiplied by an even number.

The second group of photos illustrates the rounding of the singular parts of some polyhedra which appear as buds of flowers.


Here, it is interesting to notice that the visible part of the complete flower itself (*right*) has the shape of a half octahedron.



Fig. 38 The bud of a poppy and its flower

The bud of a peony

Fig. 39

6.2.2 Paragraph 4.7 introduced a notion of foliation of a cone. This notion does not fit exactly what can be observed in nature. A better approach consists in introducing a notion of multiple protecting covering - richer than the notion of (simple) covering commonly used in mathematics.

For instance, let us consider the full 2-cone we have met in 3.2.3 (Figure 41, left), and its boundary, its associated complete hollow 1-cone, represented by the red triangle (Figure 40, middle). It is viewed as a simple covering of the full triangle. Since it has no thickness, we may cover the full triangle by any number n of replica of the hollow cone: they constitute a *multiple covering* of the full cone.

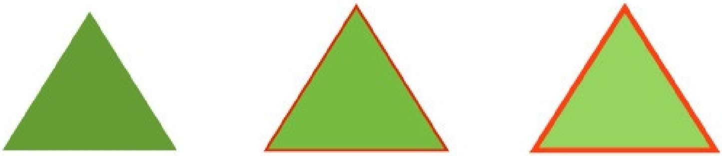


Fig. 40

Consider now a tetrahedron as a cone **C** with apex V, whose basis is the previous full triangle. Its coat C^c is a hollow 2-cone whose basis is the red triangle of Figure 42. Consider now an other hollow 2-cone with the same apex V, but whose basis is the blue triangle.

The singular points of the red basis of the given 2-cone are contained in the regular part of the blue basis of the second 2-cone. We shall call that second cone a *protecting covering* of the first one.

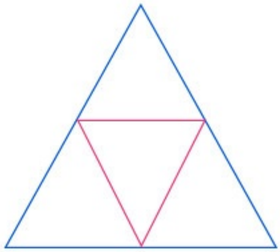


Fig. 41

Indeed the second cone is "protecting" the singular lines of the first cone.

If you iterate the process of protection of the successive 2-cones, together with a rounding of the whole construction, you get something similar to the bud of a flower characterized by an appropriate foliation.

As an example, we may choose the bud of a rose - the rose might have a $3+2$ symmetry.

Fig. 42

7 Exfoliations

In order to create new shapes, we have intensively been using attachments along singular parts. If we think in physical terms, giving some thickness to a 1, 2 ... n-dimensional domain, the k-one will be understood as less strong than the $(k+p)$ one. Thus a singular part of an object belongs in some sense to the weak part, to the most fragile part of an object.

Then the attachment of two objects along some of their singular part may show some weakness, especially if the quality of the glue or of the soldering is not the best.

That is a reason which encourages the creation of protecting coverings.

We shall call exfoliation the inverse process of creation. As it is working in the floral universe, it consists in disconnecting an object along its singular parts, through local processes of separation, of detachment.

From the metrical and physical point of view, the process of attachment is not brutal in general, but is progressive, and can be numerically controlled in time according to the point of the singular part which is reached. The operation of exfoliation has similar properties, but can be run faster than the one of creation.

Since an apex is a 0-dimensional domain, exfoliation generically begins with such singular points. If we imagine the presence of a multicoloured cloud of 1-cones, exfoliation, a big-bang coming with the vanishing of the apices gives rise to an other cloud of arms and anti-arms.

Exfoliations of polyhedra give rise to many new beautiful flowers.

8 Conclusion

The topological theory which has been presented here is fairly simple, even perhaps naive. But giving also rise to a large amount of mathematical questions, its fecundity is rather a proof of its interest. In higher dimensions, our usual mathematical tools are unable to classify singularities. We may hope that the topological approach will permit us to go further. In other respects, the construction of an algebraic topology based on cones is more complex than the classical one, but the fact that a non linear triangle remains the assembly of three 1-cones, that several ways to attach cones can be used, suggests that a finer and a richer theory could be developed. It is worth noticing that a classification of cones seems to be impossible since it includes the classification of the basis of cones, which can be cones themselves. That is why I have chosen, after the title of this article, to symbolize this theory of cones by the drawing of the snake which bites its tail.

From a pedagogical point of view, the theory is very pleasant: it is accessible to everybody, permitting the creation of a multitude of 2D and 3D cones, shapes and compositions, using modelling clay, strings, scissors, paper, pieces of cardboard, glue, and a brush. Later, software permitting, we may be able to make these constructions on computers. Using the set of these tools, an imaginative artist could have already created all the objects that have been shown on Figure 26 for example.

Via the concepts on which it stands, via the creations it allows, the theory stands in some sense at the junction of mathematics and art. Through the constructions he imagines and shapes, born of his hands, the child, the budding artist will express his dreams, and perhaps will reveal talents which will one day be expressed in an artistic activity, one of the most original of man, whether engraved in matter, or simpler and purer worked by the mind.

References

[1] C.P. Bruter, “An Introduction to the Construction of Some mathematical Objects”, in *Mathematics and Modern Art*, C.P. Bruter Ed., Springer 2012, 29-46.

[2] C.P. Bruter, “Morphologie des ensembles de bifurcation associés à des polynômes à une variable réelle. Applications.” *C.R. Acad. Sc. Paris*, t. 283 (1976), 651-654.

[3] A. Dürer (trad. Jeanne Peiffer), *Géométrie* [“*Underweysung der Messung*”], Éditions du Seuil, 1995 (*De Symmetria... and Underweysung der Messung mit dem Zirkel und Richtscheit*), 1525-1538. http://www.rarebookroom.org/Control/duruwm/index.html

[4] http://www.math-art.eu/Documents/pdfs/Cagliari2013/Cagliari_II_Bruter.pdf

Index of Authors

APÉRY François
francois.apery@uha.fr
Associate Professor (University of Mulhouse), scientific referee for the collection of mathematical models at the Institute Henri Poincaré

BRUTER Claude P.
bruter@u-pec.fr
Retired Professor (Université Paris 12). ESMA, c/o Institut Henri Poincaré, 11 rue P. & M. Curie, 75321 Paris, Cedex 05.

CADDEO Renzo
caddeo@unica.it
Professor (Università degli Studi di Cagliari), Dipartimento di Matematica e Informatica, via Ospedale 72, 09124 Cagliari, Italia.

CHACON Gerardo R.
grchacon@gmail.com
Assistant Professor, Gallaudet University, 800 Florida Avenue NE, Washington, DC 200002, USA.

COLUCCI Renato
renatocolucci@hotmail.com
Lecturer (Assistant Professor), Xi'an Jiatong-Liverpool University, 111 Ren Ai Road, Dushu Lake Higher Education Town, SIP, Suzhou, 215123, China.

DE COMITE Francesco
Francesco.De-Comite@univ-lille1.fr
Associate Professor in Computer Science (University of Lille). CRIStAL, bat M3 Info,Université des Sciences de Lille, Cité Scientifique 59655 Villeneuve d'Ascq France.

DENNER Richard
rj.denner@evc.net
Retired mathematics teacher of the secondary school. Webmaster of the ESMA since 2011.

DUNHAM Douglas
ddunham@d.umn.edu
Department of Computer Science, University of Minnesota, Duluth, MN 55812-3036, USA.

FRANZONI Gregorio
gregoriofranzoni@gmail.com
Teacher of mathematics in secondary school and university lecturer (Università degli Studi di Cagliari), Dipartimento di Matematica e Informatica, via Ospedale 72, 09124 Cagliari, Italia.

GIACARDI Livia
livia.giacardi@unito.it
Professor (Università degli Studi di Torino), Dipartimento di Matematica, Via Carlo Alberto 10, 10123 Torino.

HART George
george@georgehart.com
Stony Brook University, 3 Stony Rd. Stony Brook, NY 11790 USA.

KOZLOV Dmitri
kozlov.dmitri@gmail.com
Research Institute of Theory and History of Architecture and Town-planning, Russian Academy of the Architecture and Building Sciences21-a 7th Parkovaya St. Moscow, 105264, Russia.

LEGUIZAMON Juan S.
sebasgta@gmail.com
Pontificia Universidad Javeriana, Carrera 7 N.43-82, Bogota, Colombia.

PIU Paola
piu@unica.it
Assistant Professor (Università degli Studi di Cagliari) and co-ordinator for Sardinia of the Italian Ministry of Education programme "Piano Lauree Scientifiche", Dipartimento di Matematica e Informatica, via Ospedale 72, 09124 Cagliari, Italia.

VELICHOVA Daniela
daniela.velichova@stuba.sk
Professor of Applied Mathematics, head of the Institute of Mathematics and Physics, Slovak University of Technology, Nám. slobody 17, 812 31 Bratislava, Slovakia.