ARTISTIC FRACTAL PATTERNS

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Abstract

We describe a flexible algorithm that can create many different kinds of aesthetic fractal patterns. The main idea of the algorithm is to randomly place progressively smaller copies of a motif within a region, thus producing a fractal pattern. This has been found to work for a wide variety of shapes for both the motif and the enclosing region. We show a several examples. If the region is a fundamental region for one of the 17 "wallpaper" groups, the pattern can be extended by applying transformations from that group, giving patterns that are locally random, but have global symmetries. We also show examples of this kind of pattern.

1 Introduction

In a previous paper we described an algorithm that could fill a planar region with a theoretically infinite number of randomly placed, progressively smaller copies of a basic subpattern or *motif* [1]. This method produces a fractal pattern since the placements are random. This algorithm has proved to be quite robust and works for a wide variety of shapes for the motif and for the region. Figure 1 shows a pattern of red and blue yin-yang motifs which themselves form a global yin-yang pattern.

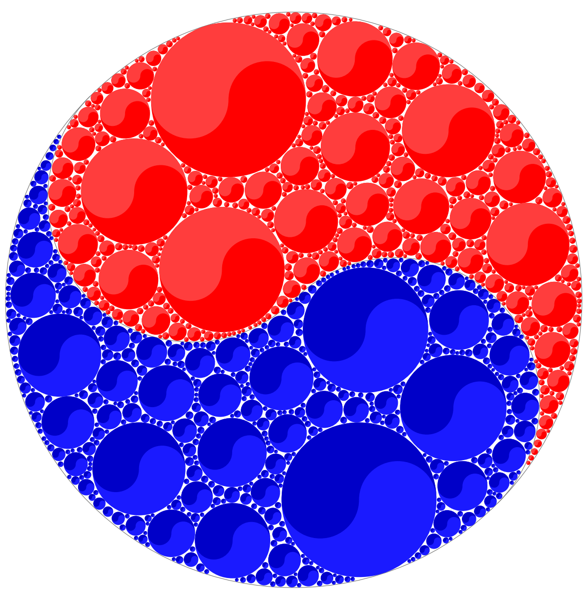

Figure 1: A fractal pattern of yin-yangs forming a yin-yang.

Subsequently we used the algorithm to fill fundamental regions of some of the 17 2-dimensionaly chrystallographic groups, also known as "wallpaper" groups. The isometries of those groups could then be applied

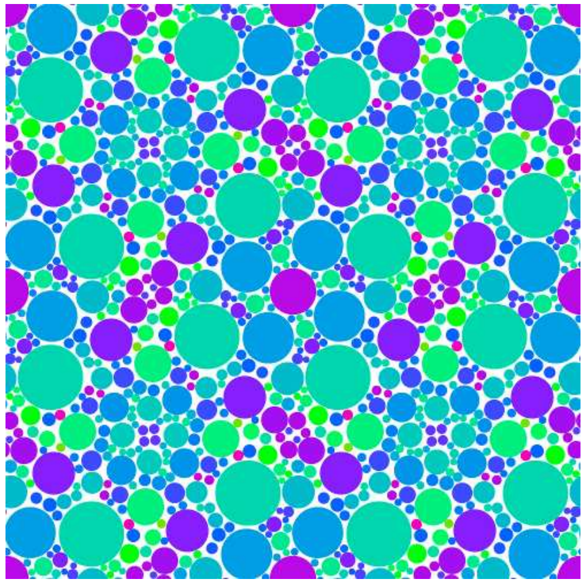

Figure 2: A locally random circle fractal with global $p4$ symmetry.

to copy that region to create a theoretically infinite pattern with those global symmetries, but was locally random [2]. Figure 2 shows such a random pattern of circles with $p4$ symmetry.

In the next section we explain how the algorithm works. Then we present a selection of different patterns that demonstrates the flexibility of the algorithm. In the next section we show some fractal wallpaper patterns. Finally, we conclude and indicate directions of future work.

2 The Algorithm

The idea of the algorithm (below) is to randomly place progressively smaller motifs m_i within a region R so that they do not overlap any previously placed motif [1]. Random positions are tried until a non-overlapping one is found. As noted in [1] for many choices of R and motifs m_i of area A_i, the following algorithm proceeds without halting:

For each $i = 0, 1, 2, \ldots$

Repeat:

Randomly choose a point within R to place the i-th motif m_i.

Until (m_i doesn't intersect any of $m_0, m_1, ..., m_{i-1}$)

Add m_i to the list of successful *placements*

Until some stopping condition is met, such as a maximum value of i or a minimum value of A_i.

It has been found experimentally by the second author that this non-halting phenomenon occurs over a wide range of choices of shapes of R and the motifs if the motifs obeyed an inverse power law *area rule*: if A is the area of R, then for $i = 0, 1, 2, \ldots$ the area of m_i, A_i, can be taken to be:

$$A_i = \frac{A}{\zeta(c, N)(N + i)^c} \tag{1}$$

where $c > 1$ and $N > 1$ are parameters, and $\zeta(c, N)$ is the Hurwitz zeta function: $\zeta(s, q) = \sum_{k=0}^{\infty} \frac{1}{(q+k)^s}$. Thus $\lim_{n\to\infty} \sum_{i=0}^{n} A_i = A$, that is, the process is space-filling if the algorithm continues indefinitely. In the limit, the fractal dimension D of the placed motifs can be computed to be $D = 2/c$ [5].

It is conjectured by the authors that the algorithm does not halt for reasonable shapes of R and m_i, and reasonable choices of c and N (depending on the shapes of R and the m_is). For this non-halting behavior, we need $N \geq 1$ and $1 < c < c_{\max}$ where $c_{\max}$ is usually less than 1.5. In fact this has been proved for $1 < c < 1.0965...$ and $N \geq 1$ by Christopher Ennis when R is a circle and the motifs are also circles [3].

3 Sample Patterns

There are many choices for an artist to make when using the algorithm to create fractal patterns: (1) the parameters c and N, (2) the shapes of R and the motifs, (3) the percentage of fill (or the number of copies of the motif), (4) the scheme for filling copies of the motif — a pattern or different colors, and (5) the orientations for copies of the motif. As a first example, Figure 3 shows a pattern of butterflies, that is butterfly motif outliines that are filled with monarch butterfly interior decorations.

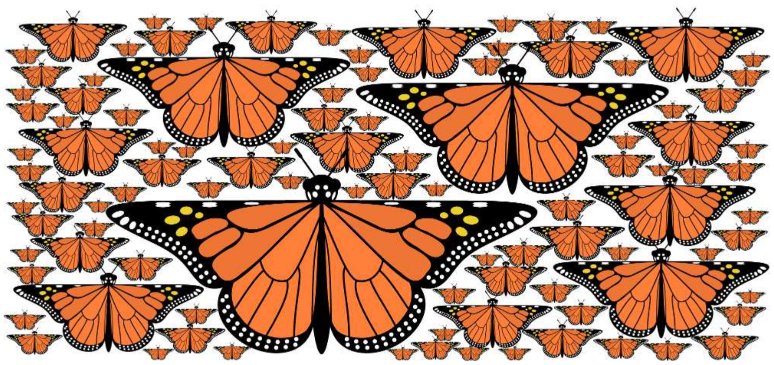

Figure 3: A butterfly pattern with 100 butterflies, c=1.26, N=1.5, and 70% fill.

We note that neither the region nor the motif needs to be simply connected or even connected. Figure 4 shows a pattern of Dali-esque non-simply-connected "eyes" that can be filled with other eyes. Figure 5 shows a pattern of $60°$–$120°$ rhombi of three orientations which are separated by $120°$, rhombi with three colors corresponding to their orientations.

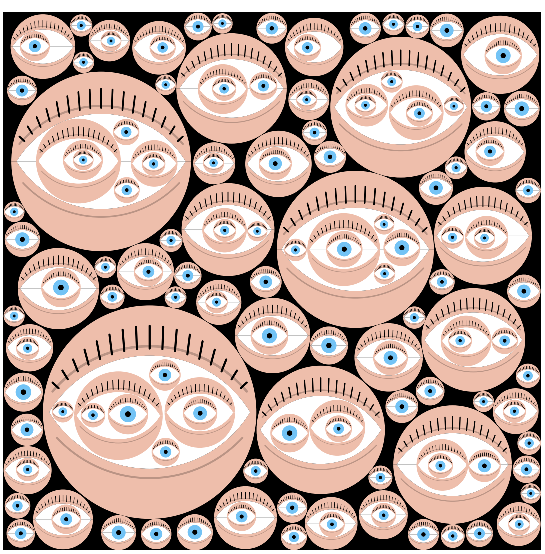

Figure 4: A pattern with "hollow" eye motifs, with 150 eyes, c=1.20, N=3, and 56% fill.

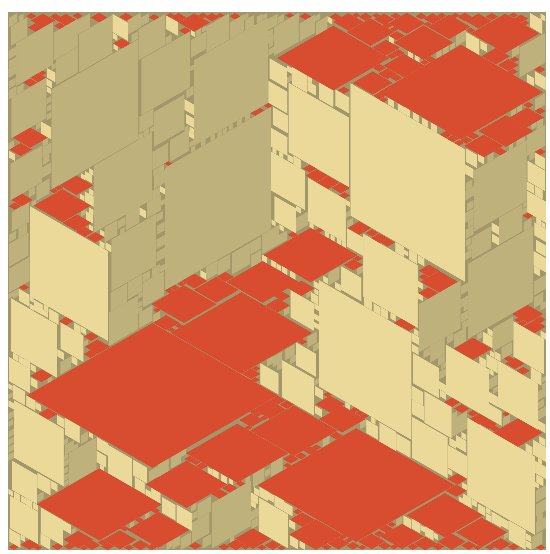

Figure 5: A pattern of three rhombi, 250 of each orientation, with c=1.52, N=8, and 91% fill.

The algorithm cycles through the three orientations. A plane tessellation by equal-sized rhombi oriented this way gives rise to the 3D Necker Cube optical illusion, which is also evident here. With this color scheme, the pattern is reminiscent of picturesque Mediterranean villages with tile roofs.

The algorithm also works for sequences of motifs — the motifs need not be the same shape as long as their areas obey the area rule. Figure 6 uses the digits 0, 1, ..., 9 as motifs, each digit receiving its own color. Again the algorithm cycles through the digits. We note that the algorithm also works for infinite sequences of "motifs", though we don't show an example. Figure 7 shows a pattern of peppers using random orientations, and random coloring independently of orientation, but but only within the gamut of the colors of hot peppers: green to yellow to orange to red.

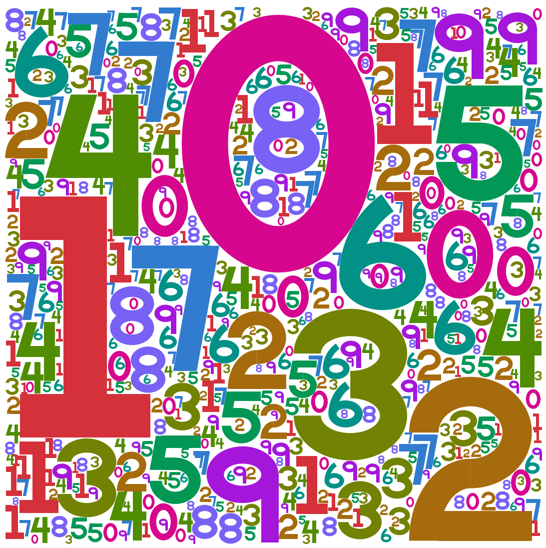

Figure 6: A sequence of 600 digit motifs 0-9, with c=1.19, N=2, and 68% fill.

Figure 7: 1200 Randomly oriented peppers with c=1.26, N=3, 80% fill.

One can observe that some of the peppers overlap the edges of the square region. In fact each one of those peppers is continued across the opposite edge: there is an orange pepper on the right edge that is continued from the left edge. There are also peppers that are continued from top to bottom. Thus this region can serve as the fudamental region for the simplest wallpaper group, $p1$. We discuss such fractal patterns with wallpaper symmetry in the next section.

4 Fractal Wallpaper Patterns

In this section we discuss "wallpaper" patterns, and fractal wallpaper patterns in particular. It has been known for over a century that there 17 different kinds of patterns that repeat in two independent directions in the Euclidean plane. Such patterns have been called *wallpaper patterns* and their symmetry groups are called plane crystallographic groups or *wallpaper groups*. In 1952 the International Union of Crystallography (IUC) established a notation for these groups, and a shorthand notation soon followed. In 1978 Schattschneider wrote a paper clarifying the notation and giving an algorithm for identifying the symmetry group of a wallpaper pattern [4].

As mentioned in the Introduction, we can create fractal wallpaper patterns by first filling a fundamental region for one of the wallpaper groups with a fractal pattern, and then extending that pattern (theoretically to the entire plane) by applying isometries from that wallpaper group. The process of filliing a fundamental region is different for each group, and thus requires a separate modification of the algorithm. Figure 7 shows a pattern of circles with $p1$ symmetry, the simplest kind of wallpaper symmetry; there are four copies of the square fundamental region.

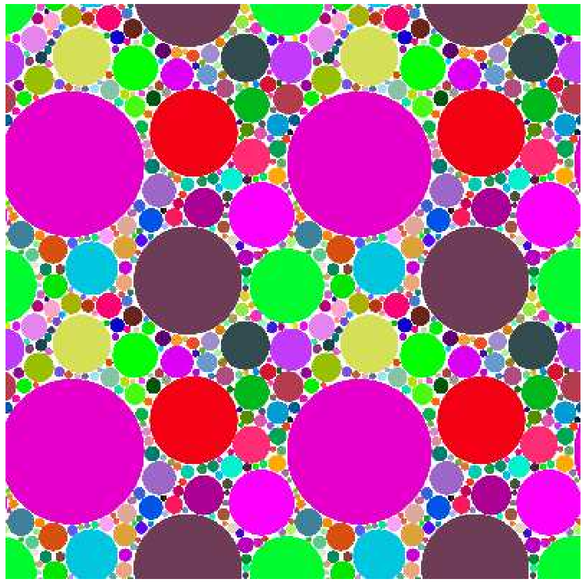

Figure 8: A circle pattern with $p1$ symmetry.

Four of the wallpaper groups have fundamental regions that are bounded by mirror (or reflection) lines. An issue that arises for these groups is what to do when a trial placement of the motif crosses a mirror boundary of the fundamental region. There are several options. The simplest option is simply to let that happen, but this leads to non-aesthic patterns with blobs of partial motifs combined on those mirrors. Another option is to simply reject such placements. This was done in the pattern of hearts (with $p2mm$ symmetry) in Figure 8 and the pattern of triangles (with $p4mm$ symmetry) in Figure 9.

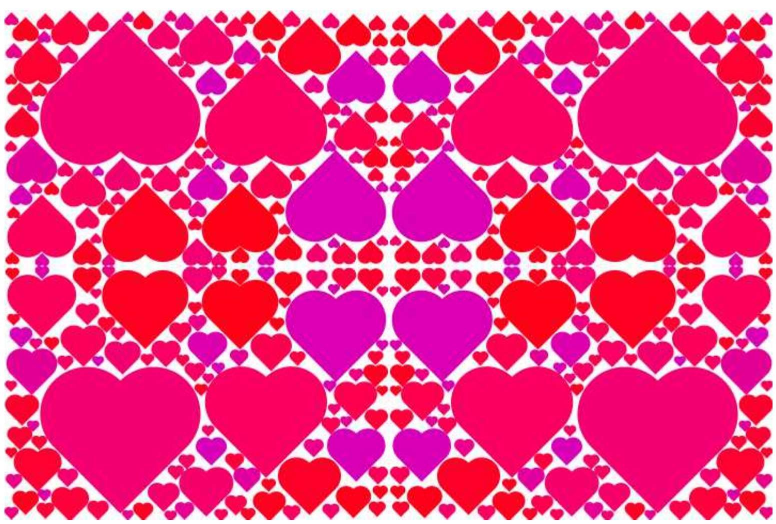

Figure 9: A random heart pattern with $p2mm$ symmetry.

Another option we devised works for motifs with at least bilateral symmetry. In this case if one of the axes of bilateral symmetry aligns with the mirror boundary, we center the motif on that mirror axis. This has been done in Figure 10, which shows a pattern of flowers with $p6mm$ symmetry.

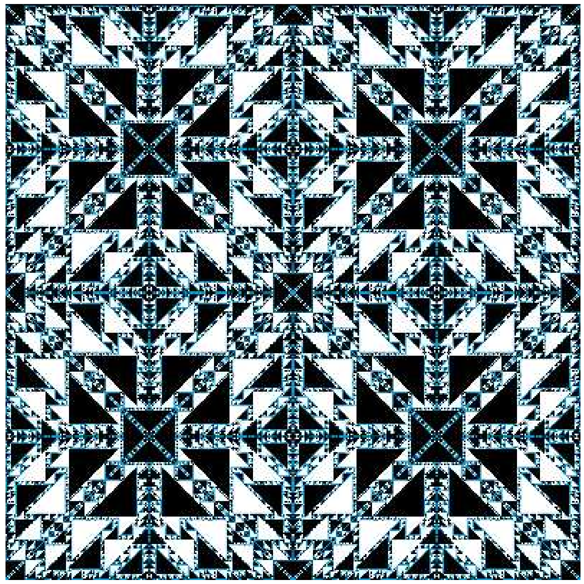

Figure 10: A pattern of black and white triangles with global $p4mm$ symmetry.

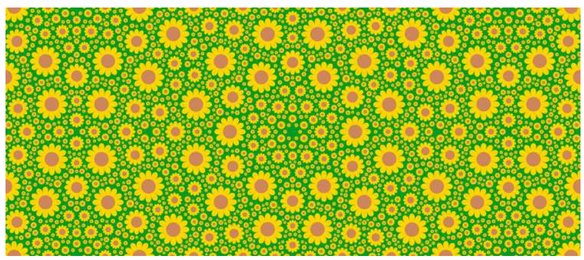

Figure 11: A flower pattern with $p6mm$ symmetry with some flowers on mirror lines.

We have also created fractal patterns with $p4$ and $p6$ symmetry. Figure 2 showed a fractal pattern of circles with $p4$ symmetry. Figure 12 shows a circle pattern with $p6$ symmetry. These symmetry groups have rotation centers: $p4$ has 2-fold and 4-fold centers, and $p6$ has 2-fold, 3-fold, and 6-fold centerss. As with the case of mirror boundaries for fundamental regions, the issue arises as to what to do when a trial placement of a motif overlaps a rotation point. Again, we could let that happen, resulting in unaesthetic blobs of overlapping motifs, or we could avoid the rotation points. But, as with the case of mirrors, we obtain an aesthetic solution by centering the motif on the rotation point if motif has such rotational symmetry. In the examples below we use circles as motifs, since they have any desired rotational symmetry.

5 Conclusion and Future Work

We have presented a very general method for creating aesthetic fractal patterns. We have extended this method to generate patterns with global wallpaper symmetry but which are locally fractal in nature. We have done this for some of the wallpaper groups, but our eventual goal is to be able to create locally fractal patterns for all 17 wallpaper groups. It would also be interesting to create corresponding spherical or hyperbolic patterns that are locally random, but have global symmetries.

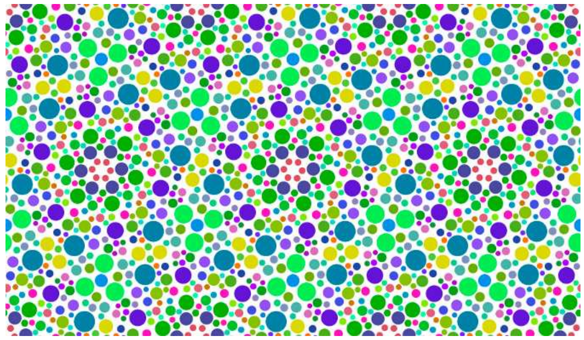

Figure 12: A circle pattern with $p6$ symmetry.

References

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[3] Christopher Ennis, *(Always) Room for One More,* MAA Math Horizons, February, 2016.

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