Knots and Links As Form-Generating Structures

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Abstract . Practical modeling of spatial surfaces is more convenient by means of transformation of their flat developments made as topologically connected kinetic structures. Any surface in 3D space topologically consists of three types of elements: planar facets (*F*), linear edges (*E*) and point vertexes (*V*). It is possible to identify the first two types of these elements with structural units of two common types of transformable systems: folding structures and kinematic nets respectively.

In the paper a third possible type of flat transformable structures with vertexes as form-generative units is considered. In this case flat developments of surfaces are formed by arranged point sets given by contacting crossing points of some classes of periodic knots and links made of elastic-flexible material, so that their crossing points have real physical contacts. A fragment of plane point surface can be reversibly converted into a fragment of a spatial surface with positive, negative or combined Gaussian curvature by means of transformation which saves connectivity between the points, but not the distances and angles between them. It was proved experimentally that this new form-generative method can be applied to modeling of both oriented and non-oriented differentiable topological 2D manifolds. The method of form-generation based upon the developing properties of periodic structures of knots and links may be applied to many practical fields including art, design and architecture.

1. Euler's Formula and Two Common Types of Kinetic Surface Models

The most general variety of geometry – topology, treats surfaces in 3D space as 2D manifolds: oriented or non-oriented. It was proved that any oriented manifold is equal to a surface of a pretzel with a some number of holes in it. The number of holes is a topological invariant called "surface genus", which is equal to zero for a sphere, one for a torus, two for a pretzel with two holes and so on. Any 2D surface can be divided into a number of polygonal meshes or facets (F) with borders or edges (E) between them, which intersect in points or vertexes (V). These three elements of an any surface are interrelated by a simple equation known as "*Euler's formula*": a number of vertexes minus a number of edges plus a number of facets is equal to two minus two multiplied by n ($V - E + F = 2 - 2n$), there n is the surface genus.

Practical modeling of 2D surfaces in 3D space is more convenient by means of transformation of their flat developments made as connected kinetic structures. There are two well-known types of such structures, which are based upon using planar (F) and linear (E) elements of surface division as their structural invariants. In the first case the result is a folding structure – a flat solid sheet divided into a number of planar facets (F) with turning linear (E) hinges between them (Figure 1). Flat kinetic folding structures are the basic form-generative principle for different types of transformable paper models, including the art of origami.

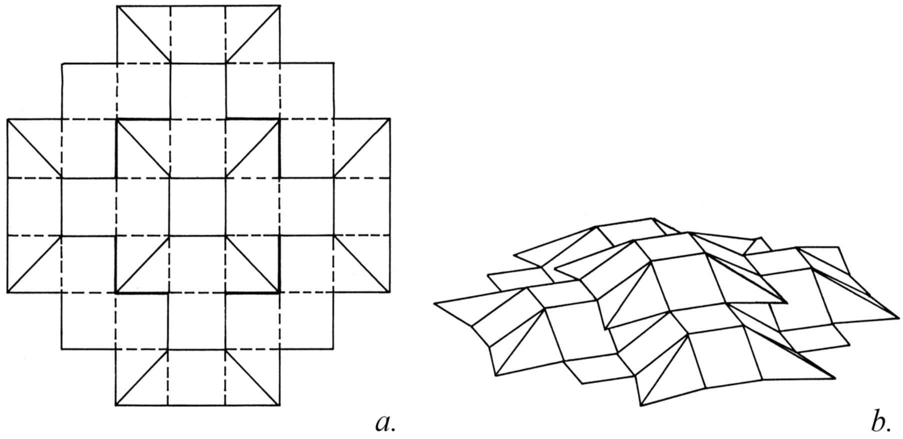


Fig. 1. *Flat folding structure as a method of form-generation of surfaces.*

In the second case the structure is a kinematic net – a flat net with non-triangle meshes assembled of linear (*E*) elements with turning point hinges (*V*) between them. The net can completely lie on a plane or be transformed fully or partly into spatial position (Figure 2). The principle of kinematic net structure has found its wide application in practical modeling of complex curved surfaces. In 1878 Russian mathematician P. L. Chebyshev stated equations for flat developments of spherical surfaces made of fabric with square meshes [1]. In the end of 19th century A. Gaudi used the method of inversion of suspended net models with the aim of form-finding in architecture. In the middle of 20th century F. Otto started his own experiments with suspended net models that lead him and his colleagues to a new approach to grid shells building theory and a whole number of architectural masterpieces [2].

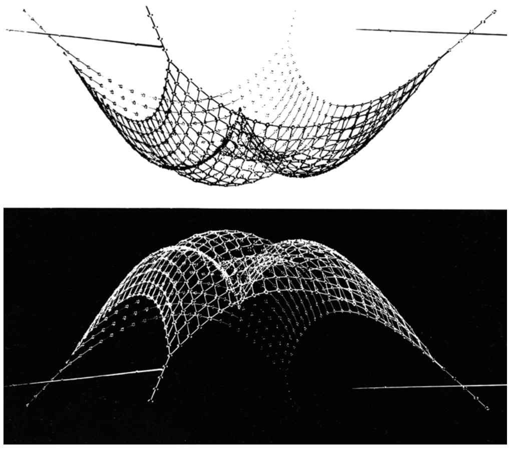


Fig. 2. *Kinematic net with square meshes as a method of form-generation of surfaces.*

2. Vertex Structures As a New Third Type of Kinetic Surface Models

In addition to the planar and linear types of flat developments of surfaces it may be proposed a third possible type of flat transformable structures with vertexes (*V*) as form-generative units. Approximations of a surface by number of points is a common method in mathematics and computer graphics. A separate point in this case is just a dot in virtual space determined by its numerical value in relation to three Cartesian coordinates.

A physical model of a point can be done as a contact of two physical bodies such as tangent solid spheres or tangent cylinders with non-parallel axis. A number of contact points on a plane or in space may be represented as a vertex or point surface (Figure 3), but to function as a transformable model of continual surface the contacting bodies must be connected between them and organized into a kinematic structure. The structure is the most important part of point models of surfaces because it coordinates behavior of great number of contact points to provide them with the possibility of synchronized sliding movement.

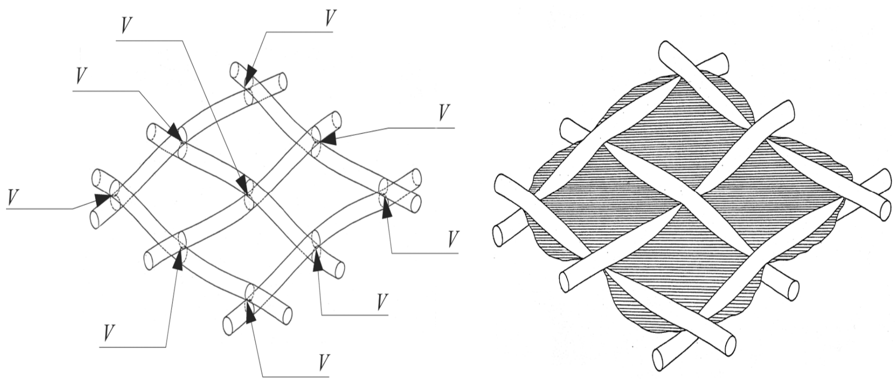


Fig.3. *A fragment of point surface made of woven resilient rods.*

This structure is not just a simple sum of neighboring kinematic units like the structures of planar (*F*) and linear (*E*) models of surfaces – it is *synergetic* in the R. B. Fuller's meaning of the word: a "behavior of integral, aggregate, whole systems unpredicted by behaviors of any of their components or subassemblies of their components taken separately from the whole" [3, p. 3].

3. Resilient Knots and Links as a Structural Principle of Vertex Surface Models

My own experimental research into different plain vertex models confirmed that the most natural forms of organizing independent point contacts into topologically connected structures are knots and links [4]. A resilient rod forms an elementary structure then its ends are joined together. As a result the rod becomes a ring – a trivial knot (Figure 4, *a*), and its structural stability depends on the ratio between the diameter of the ring and diameter of cross-section of the rod. Then the diameter of the ring is too large to resist the inner torsion forces in the bent rod, the ring turns into double nested loops (Figure 4, *b*). If the process of loops emerging is combined with joining together of the free ends of the rod, the connected rod may be knotted and take form of the simplest knot "trefoil" (Figure 4, *c*).

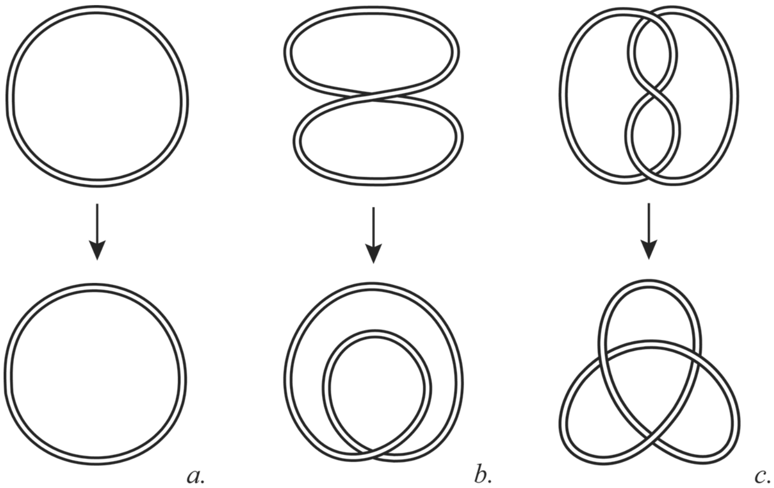


Fig. 4. *a. Resilient ring (trivial knot). b. Double loop (trivial knot). c. Simplest knot – trefoil.*

The process of "self-knotting" is very typical for long flexible-resilient strings, such as steel wire or fishing-line. Natural string-like flexible long objects, such as polymeric molecules including DNA, often take circular closed forms of rings and knots either single or linked [5]. Knots and links are widespread and natural way of structural organization for string-like flexible-resilient long objects.

4. Knots on Different 2D Surfaces

The trefoil is a "torus knot", because it can be placed without any self-crossings on the surface of a torus (Figure 5, *b*). Like a trefoil, there are knots that can be placed on the surfaces of other 2D manifolds: a ring or trivial knot on a sphere (Figure 5, *a*), "figure eight" knot – on the surfaces of pretzels with two holes (Figure 5, *c*) and so on.

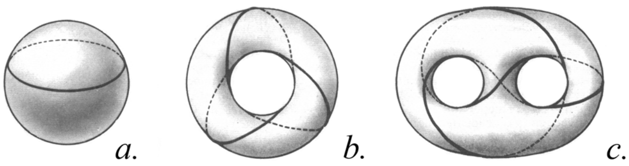


Fig. 5. *Knots on 2D manifolds.*

The trefoil knot may have two mirror types – a “left” one and a “right” one. Each of them can be tied on the torus surface without self-crossings (Figure 6, *a*, *b*), but been tied together on the same torus, they inevitably have contact points between them and form a knotted fabric on torus surface (Figure 6, *c*). If both knots made of resilient material and their crossings are really contacting, the structure will represent a model of torus point surface.

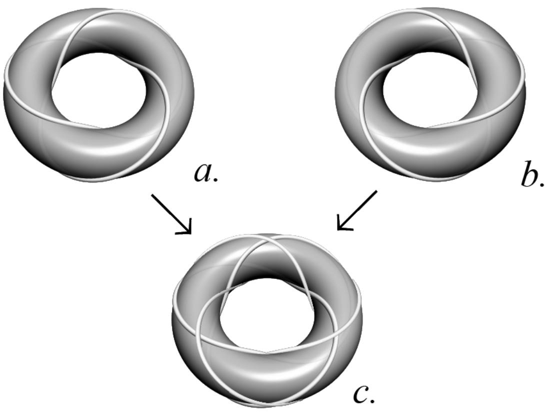


Fig. 6. *Two mirror trefoils tied together on the same torus form a torus point surface.*

The contacting points define the model of the surface – namely the exterior shape, and two mirror knots form its interior structure. In the same way it is possible to receive a point surface of an arbitrary pretzel with two mirror pretzel knots of appropriate type.

5. Energy of Resilience as Forming Principle of Cyclic Knots

Quantity of elastic energy or energy of resilience in a knotted rod depends of topological complexity of a knot and is known among other topological invariants of knots [6]. Thanks to this energy the central lines of knotted rods tend to coincide with a plain, so all their crossings tend to be really contacted, that let them form a model of flat point surface. The two mirror trefoils on the torus surface also tend to collapse, and if the torus itself disappeared, the contacting points of the two knots would place themselves in a flat ring-shaped area. And vice versa: a flat model of point surface, given by a torus knot or a link, may be transformed into a spatial state and fixed in it in order to keep the received shape.

The energy of resilience in knots also defines geometry of their structures. It force a closed resilient rod to take a shape of a ring and a rod of the same material knotted into a trefoil – a shape of a double turn coil. A coil is a natural shape for any knotted and closed resilient rod defined by its minimal internal energy of resilience. At the same time for some periodic knots [7] like trefoil, their coils may be divided into a number of equal loops or "petals": in the case of trefoil the number of loops is three. Knots of this type with natural numbers of coil turns and petal loops have a general name of "*Turk's Heads*". Simple Turk's Heads with small numbers of coil turns and petal loops made of soft non-resilient material such as rope, have a wide spread in seamen's practice as well as in the field of decoration and art [8]. It is possible to give the name of "*cyclic knots*" to the Turk's Head knots made of resilient material because of geometrical structure of their shape. From the topological point of view they are periodic closed braids [9].

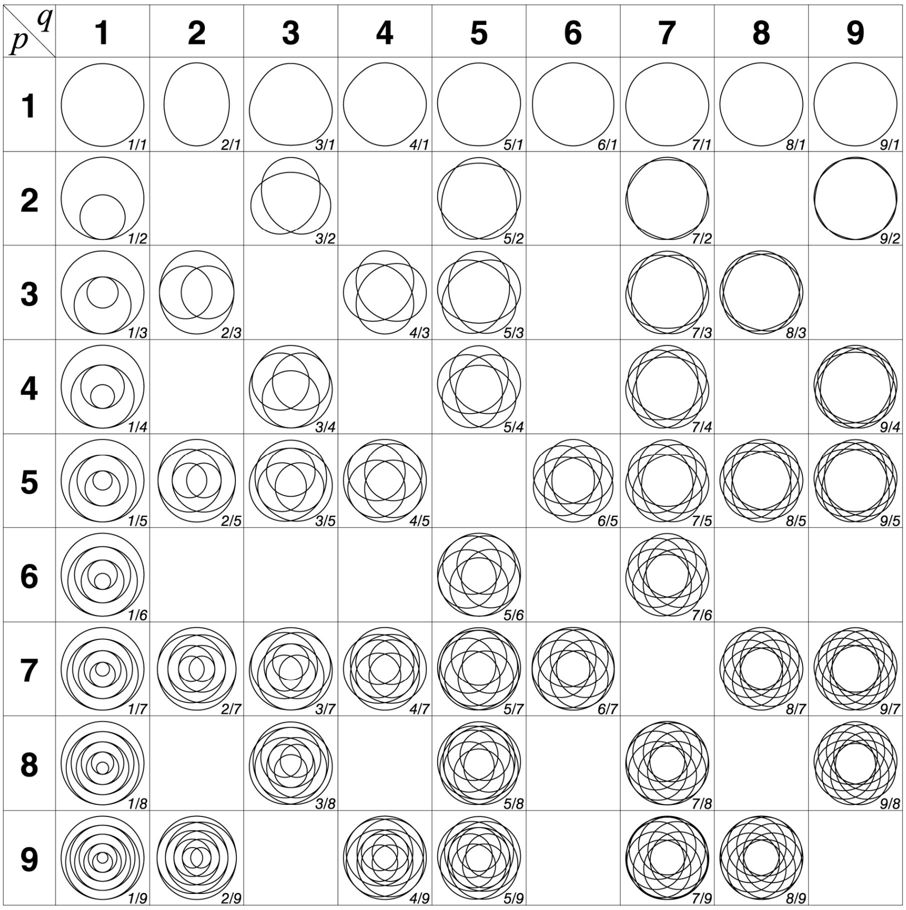


Fig.7. *Cyclic knots classified according to numbers of their turns of coils (p) and petal loops (q).*

All cyclic knots and links can be classified according to the numbers of their turns of coils (*p*) and petal loops (*q*) in the system of orthogonal coordinates. The numbers *p* and *q* may be equal to any natural number: if they are coprime – the structure is a knot (Figure 7). Here is exactly the same law as for epicycloids: diameters of their generating circles must be coprime natural numbers. If *p* and *q* are not coprime numbers the structure is a link of equivalent knots, and number of linked knots is equal to the greatest common divisor of *p* and *q*.

6. Form Generative Properties of Cyclic Knots and Links

The possibility to transform a cyclic knot from flat position to a spatial one depends of a sufficient number of its contacting crossings. This number is determined by numbers of turns of coils and petal loops, and consequently of total resilient energy of knotted rods. Increasing of the energy proportionally to the quantity of contacting crossings leads knots to a new property: from simplest knots like a trefoil they grow into complicated structures that can serve as models of point surfaces. I gave the name "*NODUS*" structures to these cyclic knots designed specially for modeling of planar and spatial point surfaces (the word "*nodus*" means "*a knot*" in Latin) [10].

A NODUS structure during its transformation changes the lengths of the edges of all its facets and angles between them. Thanks to that ability, the structure changes its geometry as a whole and creates vertex or point models of the surfaces with an arbitrary Gaussian curvature: parabolic, elliptic or hyperbolic. These three types of surfaces completely exhaust all possible internal geometries of two-dimensional manifolds [11]. As contrasted to solid models of surfaces, that can not change their Gaussian curvatures without breaks and folds, point surfaces of NODUS structures permit transition from positive Gaussian curvature (elliptic) to negative one (hyperbolic) through mediation of neutral (parabolic) curvature. The same NODUS structure can take forms of elliptic and hyperbolic curvature. A surface of torus is a combination of these two types of curvatures together with two intermediate areas of parabolic curvature (Figure 8, *a*-*c*).

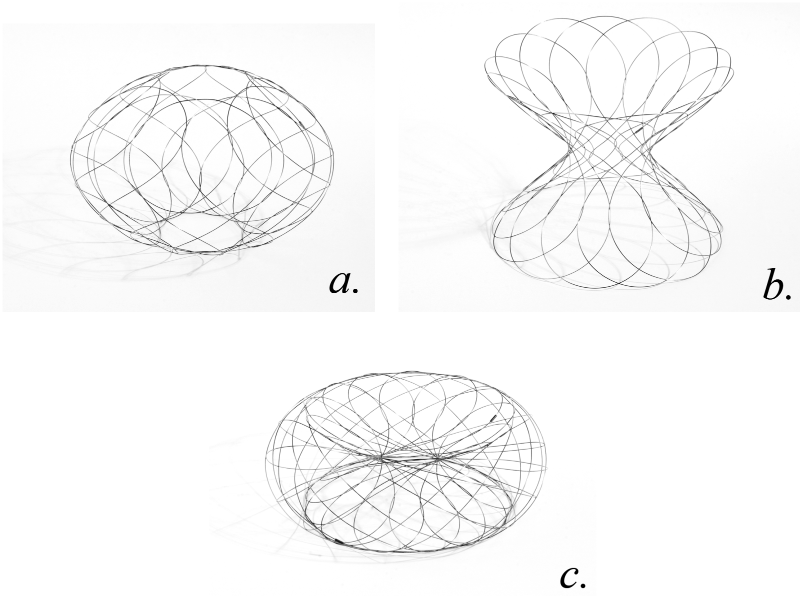


Fig. 8. *NODUS structures with elliptic, hyperbolic and combined surface curvatures.*

A surface of pretzel may be received as a combination of several torus structures (Figure 9, *a*). It is possible to create many other forms, for example surfaces with self crossings (Figure 9, *b, c*).

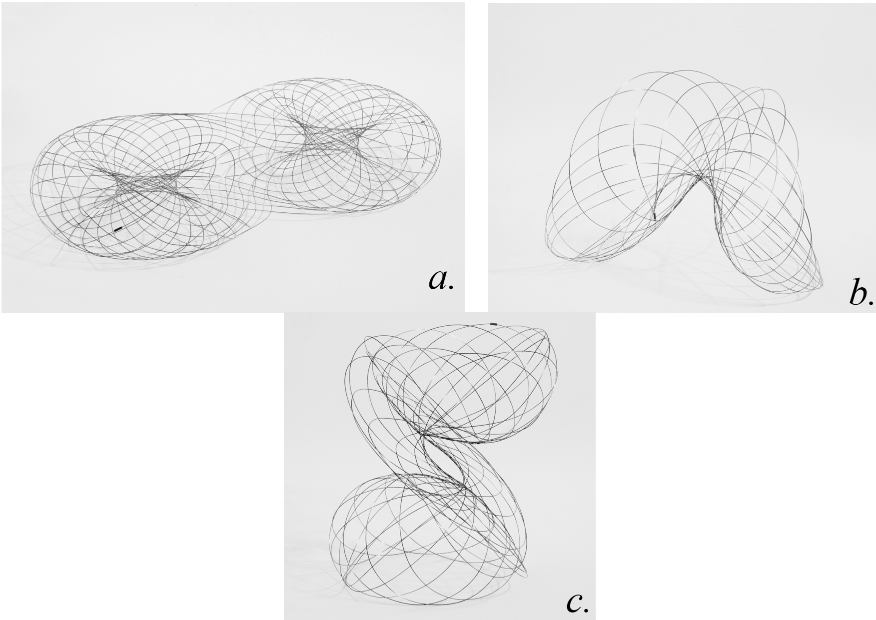


Fig. 9. *Different forms of surfaces received by means of NODUS structures.*

Also NODUS structures let make fragments of non-oriented 2D manifolds such as Möebius band with self crossing (Figure 10) or cross-cap – part of projective plane [12].

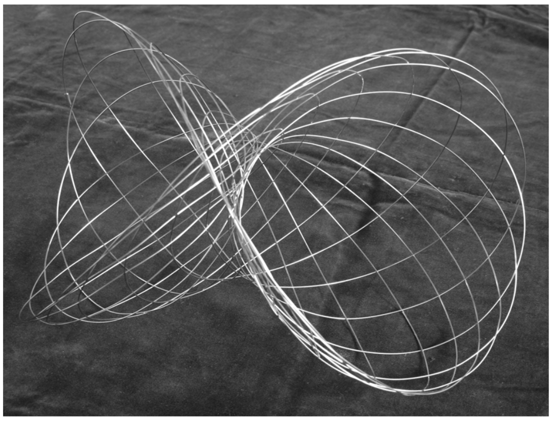
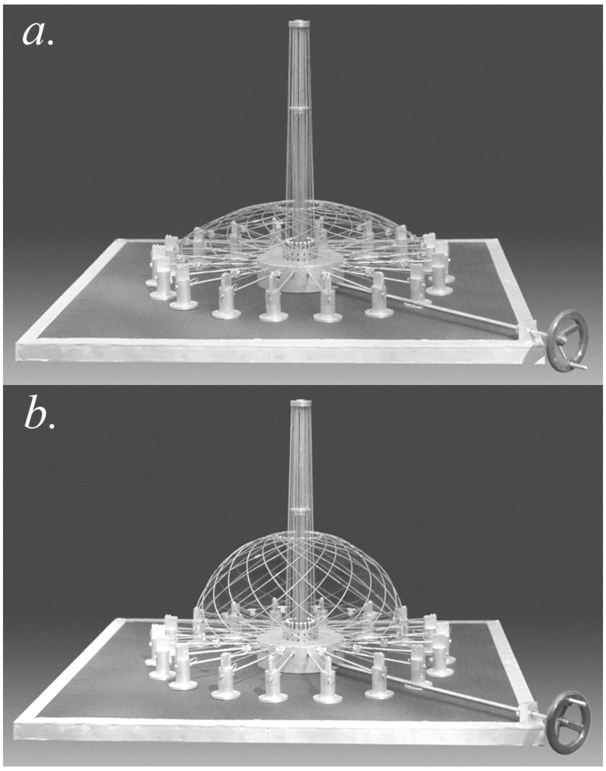


Fig. 10. *An example of one-side surface – a Möebius band with self crossing.*

Apart from the transformation of NODUS structures that changes the sign of its curvature and which can be named "*qualitative transformation*", there is another kind of transformation – the "*quantitative*" one. This transformation happens as a gradual changing of numerical value of Gaussian curvature of point surface from its minimum to a maximum value without an alteration of the curvature sign. The minimum value of Gaussian curvature may be equal to zero, and in this case the point surface of a NODUS structure approximates a piece of plane. In this case the process of transformation represents a continual sequence of changing forms, for example from spherical segment through hemisphere to sphere (Figure 11, *a*-*d*). The transformation of NODUS structure is a reversible process. Thanks to its form changing, NODUS structure accumulates elastic energy and becomes stronger. Every spatial form of NODUS structure may be strictly fixed by limitation of its mobility and as a result the transformable structure will become a stable one.



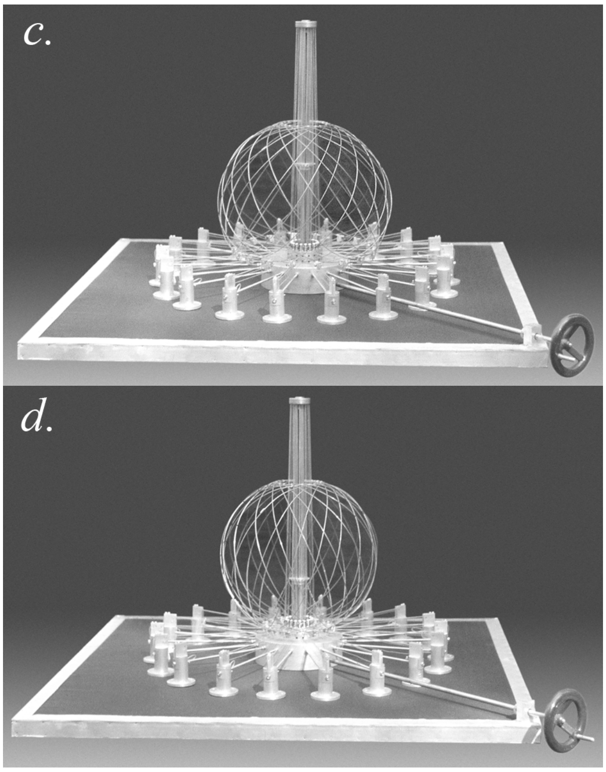


Fig. 11. *Transformation of NODUS structure as continual sequence of changing shapes.*

Polymorphous properties of NODUS structures give to an artist or a designer a suitable tool not only for finding the demanded form in space, but also for “tuning” it in environment. It is possible to envision in advance a script of development of a planar point structure into a surface in three-dimensional space by means of different dispositions of modular form-generating structures on a plane, by choice of their connections and by spatial stratifications of their contact points.

Fig. 12. *A large-sized NODUS structure in natural environment.*

According to my experiments, NODUS structures allows to extrapolate their structural properties from models to large-sized structures (Figure 12), that gives a reason to consider them also as form-generative principle for real-size kinetic architectural structures [13].

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