Mathematics and geometric ornamentation in the medieval Islamic world.

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Abstract. We discuss medieval Arabic and Persian sources on the design and construction of geometric ornaments in Islamic civilization.

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1. Introduction

Many medieval Islamic mosques and palaces are adorned with highly intricate geometric ornaments. These decorations have inspired modern artists and art historians, and they have been discussed in connection with modern mathematical concepts such as crystallographic groups and aperiodic tilings. The Islamic ornamental patterns can certainly be used to illustrate such modern notions.

Medieval Islamic civilization has also left us an impressive written heritage in mathematics. Hundreds of Arabic and Persian mathematical manuscripts have been preserved in libraries in different parts of the world. These manuscripts include Arabic translations of the main works of ancient Greek geometry such as the *Elements* of Euclid (ca. 300 BC) and the *Conics* of Apollonius (ca. 200 BC), as well as texts by medieval authors between the eighth and seventeenth centuries, with different religious and national backgrounds. In what follows I will refer to 'Islamic' authors and 'Islamic' texts, but the word 'Islamic' will have a cultural meaning only. Most 'Islamic' mathematical texts were not related to the religion of Islam, and although the majority of 'Islamic' authors were Muslims, substantial contributions were made by Christians, Jews and authors with other religious backgrounds who lived in the Islamic world.

Many Islamic texts on geometry are related to spherical trigonometry and astronomy, and most Islamic scholars who studied the *Elements* of Euclid were studying in order to become astronomers and possibly astrologers. Yet there are also Islamic works on geometrical subjects unrelated to astronomy. In almost all medieval Islamic geometrical texts that have been published thus far, one does not find the slightest reference to decorative ornaments. This may be surprising because the authors of these texts lived in the main Islamic centers of civilization and may have seen geometric ornaments frequently.

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In this paper we will see that the Islamic geometric ornaments were in general designed and constructed not by mathematician-astronomers but by craftsmen (Arabic: *ṣunnā*^{c}.) Our main question will be as follows: what kind of mathematical methods, if any, did these craftsmen use, and to what extent did they interact with mathematician-astronomers who were trained in the methodology of Greek geometry? We will discuss these questions on the basis of the extant manuscript material, which is very fragmentary. In sections 2-5 we will discuss four relevant sources, and we will draw our conclusions in the final section 6. For reasons of space, we will restrict ourself to plane ornaments and pay no attenion to decorative patterns on cupolas and to *muqarnas* (stalactite vaults).

2. Abu'l-Wafā'

We first turn to the "book on what the craftsman needs of the science of geometry"[1] by the tenth-century mathematician-astronomer Abu'l-Wafā' al-Būzjānī. This work contains some information on the working methods of the craftsmen, which will be useful for us in Section 4 below. Abu'l-Wafā' worked in Baghdad, one of the intellectual centers of the Islamic world. He dedicated his booklet to Bahā' al-Dawla, who ruled Iraq from 988 to 1012, and who apparently employed mathematicians as well as craftsmen at his court. Almost all of the booklet consists of ruler and compass-constructions belonging to plane Euclidean geometry. They are explained in the usual way, that is, by means of geometric figures in which the points are labeled by letters, but without proofs. Abu'l-Wafā' says that he does not provide arguments and proofs in order to make the subject more suitable and easier to understand for craftsmen [1, 23].

The booklet consists of eleven chapters on (1) the ruler, the compass and the *gonia* (i.e., a set square); (2) fundamental Euclidean ruler-and-compass constructions, and in addition a construction of two mean proportionals, a trisection of the angle, and a pointwise construction of a (parabolic) burning mirror; (3) constructions of regular polygons, including some constructions by a single compass-opening; (4) inscribing figures in a circle; (5) circumscribing a circle around figures; (6) inscribing a circle in figures; (7) inscribing figures in one another; (8) division of triangles; (9) division of quadrilaterals; (10) combining squares to one square, and dividing a square into squares, all by cut-and-paste constructions; and (11) the five regular and a few semi-regular polyhedra. Abu'l-Wafā' does not mention geometric ornaments.

Most of the information on the working methods of craftsmen is contained in Chapter 10. In that chapter, Abu'l-Wafā' reports about a meeting between geometers and craftsmen in which they discussed the problem of constructing a square equal to three times a given square (for an English translation see [16, 173-183]). The craftsmen seem to have had three equal squares in front of them and wanted to cut them and rearrange the pieces to one big square. The geometers

[1]Incomplete French and German versions are to be found in [21] and [20]. The complete version in Arabic is in [1] and in facsimile in [18].

easily constructed the side of the required big square by means of Euclid's *Elements*, but were unable to suggest a cut-and-paste construction of the big square from the three small squares. Abu'l-Wafā' presents several cut-and-paste methods that were used by the craftsmen, but he regards these methods with some disdain because they are approximations. Abu'l-Wafā' was trained in Euclid's *Elements* and therefore he believed that geometry is about infinitely thin lines and points without magnitude, which exist in the imagination only. He complains that the craftsmen always want to find an easy construction which seems to be correct to the eyesight, but that they do not care about a proof by what Abu'l-Wafā' calls "the imagination." He declares that the constructions that can be rigorously proven should be distinguished from approximate constructions, and that the craftsmen shoud be provided with correct constructions so that they do not need to use approximations anymore.[2] We do not know how the booklet was received but the 16th-century Persian manuscript which we will study in Section 4 contains a rich variety of approximate constructions.

3. The Topkapı Scroll

The craftsmen themselves seem to have left us with very few documents about their activities in the field of geometric ornamentation. The most important published example is the so-called Topkapı Scroll, which is now preserved in the Topkapı Palace in Istanbul, and which has appeared in the magnificent volume [14]. This 29.5 m long and 33 cm wide paper scroll is undated and may have been compiled in Northwestern Iran in the 16th century, but the dating is uncertain. The scroll consists of diagrams without explanatory text. Many of these diagrams are related to calligraphy or muqarnas and therefore do not concern us here. Some of the diagrams concern plane tilings. I have selected one non-trivial example in order to draw attention to the characteristic (and frustrating) problems of interpretation. The drawing on the scroll [14, p. 300] consists of red, black and orange lines, which are indicated by bold, thin and broken lines respectively in Figure 1 (for a photo of the manuscript drawing see also [17]). The broken lines in Figure 1 define a set of five tiles, called *gireh*-tiles in the modern research literature, from the Persian word *gīreh*, which means knot. The thin lines form a decorative pattern which can be obtained by bisecting the sides of the gireh-tiles, and by drawing suitable straight line segments through the bisecting points. It is likely that the pattern was designed this way, but one cannot be sure because the scroll does not contain any explanatory text. The gireh tiles of Figure 1 have drawn recent attention because they can be used to define aperiodic tilings. In the absence of textual evidence, it is impossible to say whether the craftsmen had an intuitive notion of aperiodicity (for a good discussion see [8]).

[2]Note that Abu'l-Wafā' presents an approximate construction of the regular heptagon by ruler and compass. Just like many of his Islamic contemporaries, he probably believed that the regular heptagon cannot be constructed by ruler and compass.

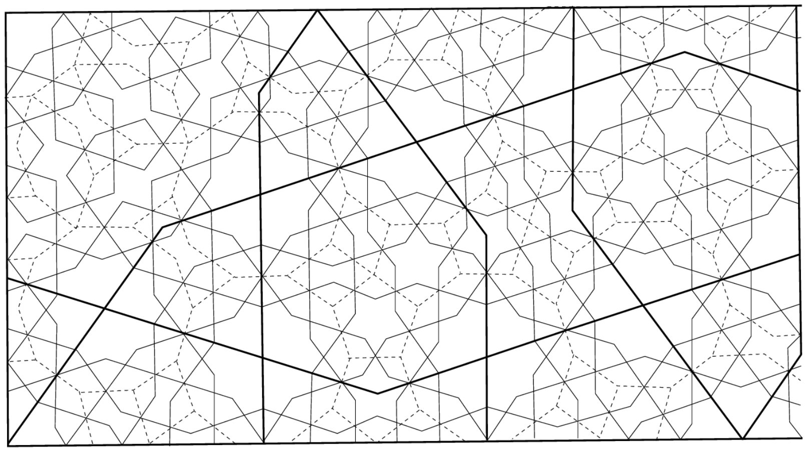

Figure 1. Drawing by Dr Steven Wepster.

4. An anonymous Persian treatise

One would like to have a medieval Islamic treatise, written by a craftsman, in which the design and construction of ornaments is clearly explained. Such a treatise has not been found, and thus far, only a single manuscript has been discovered in which diagrams on geometrical ornaments are accompanied by textual explanations. In this section we will discuss what this manuscript can tell us about the main question in the beginning of the paper. The manuscript is a rather chaotic collection of 40 pages of Persian text and drawings (for some photos see [14, 146-150]). The text consists of small paragraphs which are written close to the drawing to which they refer, and although the texts and drawings appear in a disorganized order and may not be the work of a single author, I will consider the collection as one treatise.[3] It may have been compiled in the sixteenth century, although some of the material must be older as we shall see.

The treatise belongs to a manuscript volume of approximately 400 pages [5, 55-56]. Some of the other texts in the manuscript volume are standard mathematical works such as an Arabic translation of a small part of Euclid's *Elements*. But the treatise itself does not resemble a usual work by a mathematician or astronomer in the Islamic tradition. I believe that the treatise is the work of one or more craftsmen because it agrees with most of what Abu'l-Wafā' says about their methodology. The treatise provides much additional information on the working methods of the craftsmen and it also shows that they were really involved with the design and construction of geometrical ornaments. In order to illustrate these points, I have selected the following four examples 4.1 through 4.4 from the treatise.

[3]The treatise was translated into Russian [6, 315-340] and modern Persian [2, 73-93], and a full publication of it with English translation was planned by Alpay Özdural (cf. [15]), who unfortunately passed away in 2003 before he completed the project. The Persian text is scheduled to be published, with translation and commentary, by an interdisciplinary research team in 2013.

4.1. The treatise contains many approximation constructions, including a series of ruler-and-compass constructions of a regular pentagon by means of a single compass-opening. In these constructions, the compass opening is assumed to be either the side of the required regular pentagon, or the diagonal, the altitude, or the radius of the circumscribing circle. Here is one such construction with my paraphrase of the manuscript text [12, 184b]. Figure 2 is a transcription of the figure in the manuscript, in which the labels (the Arabic letters *alif*, *bā'*, ...) are rendered as A, B, ..., and Hindu-Arabic number symbols are represented by their modern equivalents. The Persian text says:

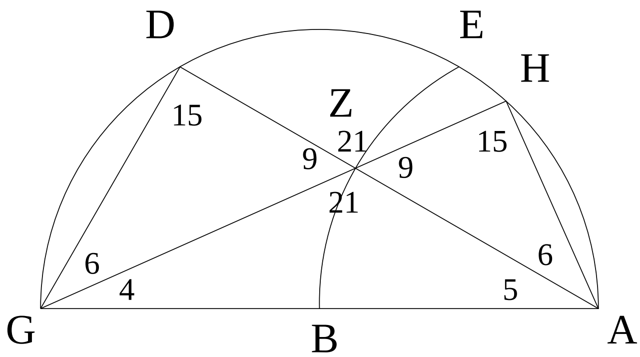

Figure 2

"On the construction of *gonia* 5 by means of the compass-opening of the radius, from *gonia* 6. On line AG describe semicircle ADG with center B. Then make point A the center and describe arc BE. Then make point G the center and on the circumference of the arc find point D and draw line AD to meet arc EB at point Z. Draw line GZ to meet the circumference of the arc at point H. Join lines AH, GH.[4] Each of the triangles AZH, GZD is *gonia 5*, and the original triangle ADG was *gonia 6*, ..."

Points A, E, D and G are four angular points of a regular hexagon, and DH is the side of the regular pentagon inscribed in the same circle. The construction is a good approximation,[5] but it is not exact so Abu'l-Wafā' would not have approved it. In Chapters 3 and 4 4 of his booklet, Abu'l-Wafā' provided exact constructions of the regular pentagon using a fixed compass-opening. The *gonia* is mentioned by Abu'l-Wafā' as an instrument used by craftsmen. From the Persian treatise we infer that *gonia* n is a set square with angles 90^o, $\frac{180}{n}^o$ and $90 - \frac{180}{n}^o$. In Figure 2, angles are expressed in units such that 15 units are a right angle. In the Islamic tradition, the division of the right angle into 90 degrees, subdivided sexagesimally, was only used in mathematical astronomy and mathematical geography.

[4]Instead of GH the manuscript says incorrectly DH.

[5]This is easily shown by modern elementary geometry. Suppose that the radius of the circle is 1, and drop a perpendicular ZP onto AG. Then $ZA = 1, \angle ZAP = 30^o, ZP = \frac{1}{2}, AP = \frac{\sqrt{3}}{2}, GP = 2 - \frac{\sqrt{3}}{2}, \angle ZGP = \arctan \frac{ZP}{GP} \approx 23.8^o$. Because $\angle DGP = 60^o$, $\angle ZAH = \angle ZGD \approx 36.2^o$.

4.2. Abu'l-Wafā' says that the craftsmen are interested in cut-and-paste constructions, and the Persian treatise contains many such constructions. Some of these are explained by one or more paragraphs of text, but the following example is presented without accompanying text.

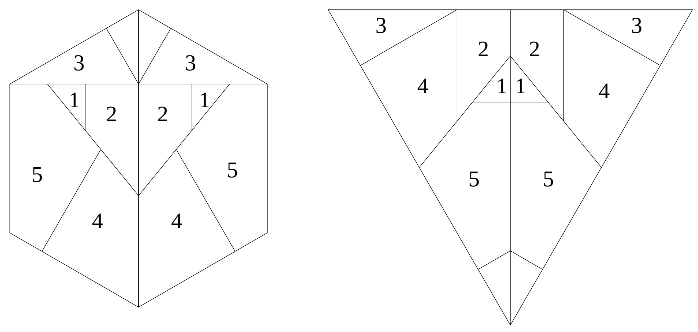

Figure 3

Figure 3 displays a regular hexagon and an isosceles triangle, dissected into pieces such that both figures can be composed from these pieces. Figure 3 is derived from the manuscript [12, 197a] with the difference that I have arbitrarily assumed the isosceles triangle to be equilateral, and I have drawn the figure in a mathematically correct way. In the manuscript, the pieces are indicated by numbers (as in Figure 3) so the correspondence is clear. Since there is no text in the manuscript, the reader does not have a hint how exactly the pieces have to be cut. I invite the reader to work out the details for himself. After this exercise, she or he will probably be convinced that the manuscript was intended to be used under the guidance of a competent teacher who could provide further information. It should be noted that the pieces no. 1 and 2 in the manuscript are drawn in such a way that no. 1 is wider than no. 2. This may happen if the vertex angle of the isosceles triangle is less than 54^o; figure 4 has been drawn for a vertex angle of $\frac{360}{7}^o$. It is tempting to assume that the craftsmen had a general dissection of an isosceles (rather than an equilateral) triangle in mind, but because there is no accompanying text, one cannot be sure. The construction is mathematically correct but there are also approximate cut-and-paste constructions in the Persian treatise.

It is not necessary to assume that the fancy cut-and-paste construction of Figure 3 and 4 was used in practice. Just like European arithmetics teachers in later centuries, Islamic craftsmen may have challenged one another with problems which surpassed the requirements of their routine work.

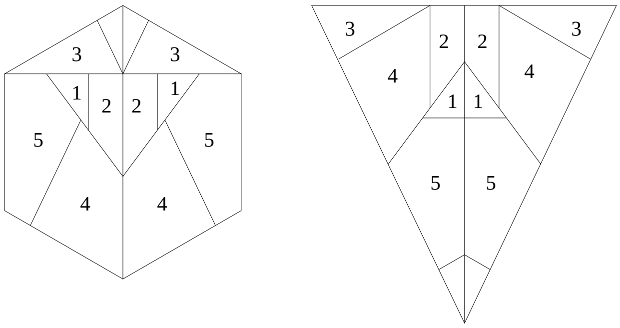

Figure 4

4.3. The many drawings of geometric ornaments in the Persian treatise show that its authors were deeply involved with the design and construction of ornamental patterns. I have selected an example which is also found on a real building, namely the North Cupola of the Friday Mosque in Isfahan, which was built in the late eleventh century. The Persian text laconically introduces the ornamental pattern as follows ([12, 192a], [14, 148]) with reference to Figure 5.[6]

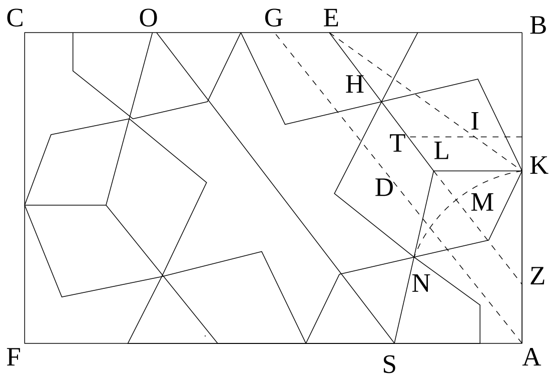

Figure 5

“Make angle BAG three sevenths of a right angle. Bisect AG at point D. Cut off BE equal to AD. Produce line EZ parallel to AG. Draw line TI[7] parallel to BE, bisect TE at point H, and make TI equal to TH. Extend EI until it intersects AB at point K. Produce KL parallel to BE. With center Z draw circular arc KMN in such a way that its part KM is equal to MN. On line AF take point S and that is the center of a heptagon. Complete the construction, if God Most High wants.

Or construct angle ELN equal to angle ELK and by means of line LN find the center S.

[6] Broken lines in Figure 5 also appear as broken lines in the manuscript.

[7] The text does not make clear that T is an arbitrary point on segment EZ.

Or cut off EO equal to EL, so that O is the center of a heptagon. And make line OS parallel to GA and equal to AG[8] Then point S is the center of another heptagon. Or else let GO be equal to AS. God knows best."

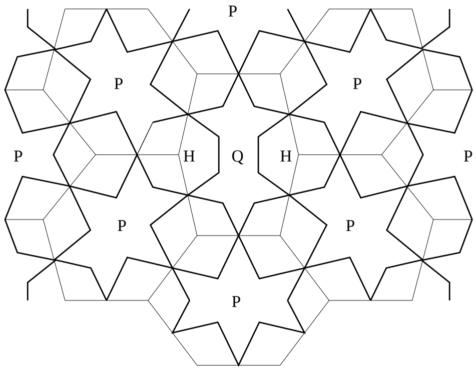

Figure 6

The text does not inform the reader what should be done with the completed figure. Apparently the rectangular figure in the manuscript and its mirror image should be repeated as suggested by figure 6. Thus one obtains the pattern in the north cupola of the Friday Mosque.[9]

The pattern can be linked to *gireh* tiles such as in Figure 1 above. These *gireh* tiles are not mentioned explicitly in the Persian treatise; all information in the treatise about figure 5 is contained in the passages quoted above. Let $\alpha = \frac{1}{7} \times 180^{\text{o}}$ and take as gireh tiles two types of equilateral hexagons with equal sides (thin lines in figure 6), of type P with angles $4\alpha, 5\alpha, 5\alpha, 4\alpha, 5\alpha, 5\alpha$, and of type Q with angles $4\alpha, 4\alpha, 6\alpha, 4\alpha, 4\alpha, 6\alpha$. Now draw suitable lines through the midpoints of the sides, in such a way that the "stars" inscribed in P and Q emerge, with angles 2α at the midpoints of the sides of the gireh tiles. The heptagons H in figure 6 are regular. Pattens with regular heptagons are rarely found on Islamic buildings so the pattern in the manuscript and on the North Cupola probably go back to the same designer or designers. The pattern on the North Cupola of the Friday Mosque consists of the thick lines in Figure 6 with some additional embellishments but without the *gireh* tiles in Figure 6.

4.4. My fourth and final example from the Persian treatise will reveal some information about the relationship between craftsmen and Islamic mathematician-astronomers who had been trained in Greek mathematics. As an introduction, consider a pattern from the Hakim Mosque in Isfahan (Figure 7). The pattern is inspired by a division of a big square into a small square and four kites.[10] Two of the angles of each of the kites are right angles.

[8]The manuscript has AD by scribal error.

[9]For a photograph see [9].

[10]See [7]. The pattern is inscribed with calligraphy: Allāh in the central square and Muḥammad and ᶜAlī in the four kites.

Figure 7

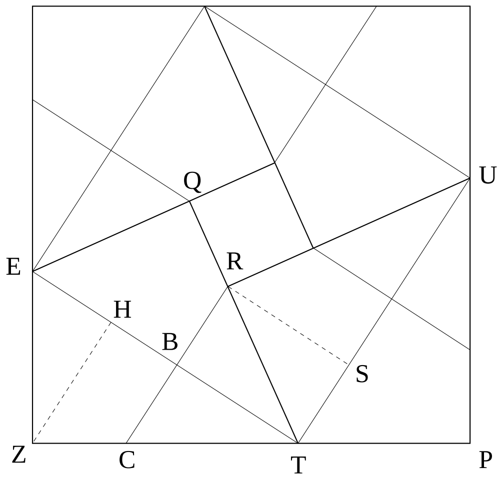

Figure 8

Figure 8 is a partial transcription of a figure in the Persian treatise [12, 189b], but the labels and broken lines are my own additions.[11] The figure displays a big square with side ZP, subdivided into a small square with side RQ, and four big kites such as $EQTZ$ and $RTPU$, each with two right angles, and with pairwise equal sides ($QE = EZ$, $QT = TZ$, $RT = TP$, $RU = UP$). Note that the four longer diagonals of the big kites also form a square with side ET, which I call the intermediate square. In the special case of Figure 8, the side QR of the small square is supposed to be equal to the distance RB between each angular point of the small square and the closest side of the intermediate square. Then each big kite such as $EQTZ$ can be divided into two right-angled triangles BRT, BCT, and two small kites such as $EQRB$, $EBCZ$ with two right angles and pairwise equal

[11] I have labelled the points in Figure 8 to highlight the correspondence with Figure 10 below.

sides ($EQ = EB$, $RQ = RB$, $EB = EZ$, $CB = CZ$). Thus we have four big kites and eight small kites, and for easy reference, I will call the resulting division of the big square the twelve kite pattern.

Almost a quarter of the Persian treatise is somehow devoted to the twelve kite pattern. If we draw perpendiculars ZH and RS to ET and TU respectively, $ZH = RS = BT$. The two sides EZ and EB of the small kite $EBZC$ are also equal, so in the right-angled triangle EZT we have $ZH+EZ = ET$. The twelve kite pattern can be constructed if a right-angled triangle (such as EZT) can be found with the property that the altitude (ZH) plus the smallest side (ZE) is equal to the hypotenuse (ET). The text states that "Ibn-e Heitham" wrote a treatise on this triangle and constructed it by means of two conic sections, namely "a parabola and a hyperbola". No further details are given, and no conic section is drawn anywhere in the Persian treatise. But the text contains a series of approximation constructions of the twelve kite pattern, such as the following [12, 189b] (Figure 9). The text reads:

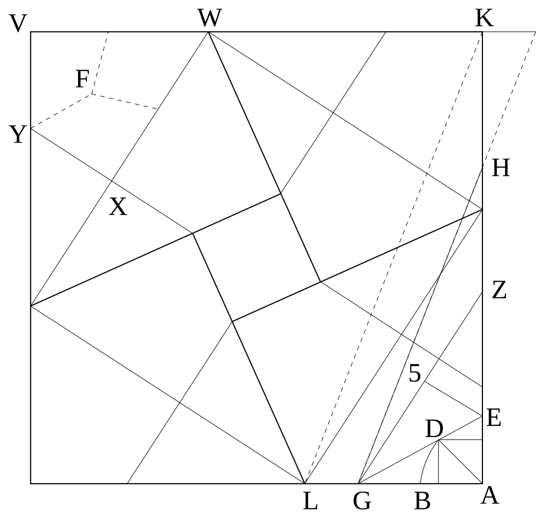

Figure 9

"Line AD is the diagonal of a square. The magnitudes of AB, BG are equal and AD is equal to AB. Find point E on the rectilinear extension of line GD. Then each of EZ, ZH is equal to AG. Join line GH and through point K draw line KL parallel to GH. Find point L, the desired point has now been obtained."

The approximation is sufficiently close for all practical purposes: if the side of the square is 1 meter, the difference between the correct and approximate positions of L is only a few millimeters.[12] It does not follow that the approximation presupposes a deep mathematical knowledge. In the figure in the manuscript, the eight

[12]If the side of the "square" in the beginning is set equal to 1, we have $AD = \sqrt{2}, AG = 2\sqrt{2}$, $\frac{AE}{AG} = \frac{1}{2\sqrt{2}-1}$ so $AE = \frac{1}{7} \cdot (8+2\sqrt{2}), AZ = \frac{1}{7} \cdot (8+16\sqrt{2}), \angle ZGA \approx 57.12\ldots \approx 57^{o}7'$. Note that $\angle ZGA$ in figure 9 corresponds to $\alpha = \angle ZET$ in Figures 8 and 10.

small kites are all subdivided into three even smaller kites with pairwise equal sides and at most one right angle. In Figure 9 the subdivision is indicated by broken lines in only one kite $VWXY$ (these labels are mine) in the upper left corner. One may guess that $FV = \frac{1}{2}VW$ and note that F is located on the bisector of angle WVY. The first step of the approximation boils down to the construction of a triangle ADG similar to VFW.

For further details on the Persian treatise we refer to the planned edition with translation and commentary which is scheduled to appear in 2013.

5. Mathematicians on the twelve kite pattern

The reference to "Ibn e-Heitham" in the Persian treatise shows that the twelve kite pattern was also studied by mathematician-astronomers. We will now discuss what is known about these studies because they will give us some further hints about the interactions between mathematician-astronomers and craftsmen. "Ibn e-Heitham" is a Persian form of Ibn al-Haytham (ca. 965-1041), a well-known Islamic mathematician-astronomer who was interested in conic sections. His treatise on the twelve kite pattern has not been found but one of the extant works of the famous mathematician-astronomer and poet ^{c}Umar Khayyām (1048-1131) is also of interest here. The work is written in Arabic and entitled "treatise on the division of a quadrant". It begins in the following uninspiring way (Figure 10, [10, 73]): "We wish to divide the quadrant AB of the circle $ABGD$ into two parts at a point such as Z and to draw a perpendicular ZH onto the diameter BD in such a way that the ratio of AE to ZH is equal to the ratio of EH to HB, where E is the center of the circle and AE is the radius." Khayyām does not give the slightest indication of the origin or relevance of this problem. He draws the tangent to the circle at Z, which tangent intersects BE extended at T, and he shows that in the right angled triangle EZT, the sum of the altitude ZH plus the shortest side ZE is equal to the hypotenuse ET.[13] Thus the problem is inspired by the twelve kite pattern, but Khayyām does not mention the relationship with this pattern or with geometric ornamentation in general. In a new figure (not rendered here), Khayyām puts, in the notation of Figure 10, $EH = 10$ and $ZH = x$, so $ZE = \sqrt{100 + x^2}$ and by similar triangles $HT = \frac{x^2}{10}$. He then shows that the property $ZH + EZ = ET$ boils down to the cubic equation $x^3 + 200x = 20x^2 + 2000$, or in a literal translation of his words: "a cube and two hundred things are equal to twenty squares plus two thousand in number" [10, 78]. He then proceeds to construct a line segment with length equal to the (positive) root x of this equation by the intersection of a circle and a hyperbola. An anonymous appendix [10, 91] to Khayyām's text contains a direct construction of point Z in figure 10 as a point of intersection of the circle and the hyperbola through point B whose asymptotes are the diameter

[13] Proof: In Figure 10 by similar triangles $EH : EZ = EZ : ET$, and because $EZ = EB$ we have $EH : EB = EB : ET$ and therefore $EH : (EB - EH) = EB : (ET - EB)$, that is to say $EH : HB = EB : BT$. By assumption $EH : HB = AE : ZH$ so because $AE = BE$ also $EH : HB = EB : ZH$. We conclude $ZH = BT$, so $EZ + ZH = EB + BT = ET$.

AEG and the tangent GM (broken lines in figure 10). None of this was relevant to a craftsman who wanted to draw the twelve kite pattern, and Khayyām declares that numerical solutions of the cubic equation could not be found. In order to find a numerical approximation of arc ZB, Khayyām rephrases the problem about the quadrant in trigonometrical form as follows: to find an arc such that "the ratio of the radius of the circle to the sine of the arc is equal to the radius of the cosine to the versed sine." In modern terms, if $\alpha = \angle ZET$ and the radius is 1, the ratio $AE : ZH = EH : BH$ is equivalent to $1 : \sin\alpha = \cos\alpha : (1 - \cos\alpha)$. Khayyām says that this problem can be solved by trial and error using trigonometrical tables and that he found in this way $\alpha \approx 57^{o}$, and if $AE = 60$ then $ZH \approx 50$, $EH \approx 32\frac{2}{3}$ and $BH \approx 27\frac{1}{3}$. He also says that one can solve the problem more accurately. Using the trigonometrical tables that were available in his time, he could have computed the required arc in degrees and minutes by linear interpolation.[14] This information on sexagesimal degrees and minutes may not have been of much use to craftsmen as we have already seen in 4.2 above. We may also compare with a reference by the Iranian mathematician and astronomer Al-Bīrūnī (976-1043) in a work on the qibla (direction of prayer towards Mecca). Al-Bīrūnī computes the qibla at Ghazni (Afghanistan) by trigonometrical methods as 70 degrees and 47 minutes West of the South point on the local horizon. He then adds a ruler-and-compass approximation construction for "builders and craftsmen," who "are not guided by degrees and minutes" ([4, 286], compare [3, 255-256]).

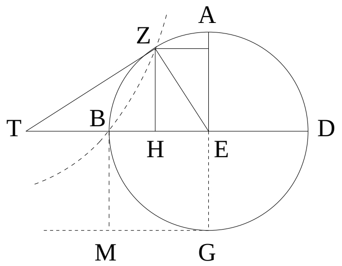

Figure 10

6. Conclusion

We now return to the main question in the introduction to this paper. Because the evidence is so scarce, it is not clear to what extent we are able to generalize the information which we can obtain from the available manuscript sources. But if this can be done, the following may be suggested about the main differences

[14] If we use modern methods and put $x = \tan\alpha$, we have $HZ = 10x$ if $HE = 10$. so $10x$ is a root of Khayyām's cubic equation, and therefore $x^3 + 2x = 2x^2 + 2$. The equation is irreducible over the rational numbers, so the twelve kite pattern cannot be constructed by ruler and compass. The equation has one real root $x = 1.54369\ldots$ so $\alpha \approx 57.06^{o} \approx 57^{o}4'$.

between Islamic craftsmen who designed and constructed ornaments, and Islamic mathematician-astronomers who were trained in Greek geometry:

- mathematician-astronomers worked with geometric proofs in the style of Euclid's *Elements*. Craftsmen were familiar with the Euclidean way to draw figures, using letters as labels of points (but also the number 5 in Figure 9 above). Craftsmen did not use geometric proofs and they had not been trained in the methods of Euclid's *Elements*.

- Texts written by mathematician-astronomers usually contain sufficient explanation to understand the mathematics. An oral explanation is not absolutely necessary. Texts and diagrams by craftsmen are often ambiguous, and oral explanations were essential.

- mathematician-astronomers distinguished between exact and approximate geometrical constructions. Craftsmen did not distinguish between these constructions if the result was acceptable from a practical point of view.

- Craftsmen used some geometrical instruments not found in the theoretical works of Greek geometry, such as a set-square and a compass with fixed opening.

The following relationship between craftsmen and mathematicians may be suggested. Mathematicians such as Ibn al-Haytham and ^{c}Umar Khayyam may have regarded the designs of craftsmen as hunting ground for interesting mathematical problems. Thus the twelve kite pattern led to construction by means of conic sections, as in figure 10 above. These constructions were a favorite research topic in the tenth and eleventh century among Islamic mathematicians who had studied the *Conics* of Apollonius (ca. 200 BC). However, Khayyām did not reveal that his geometric construction problem was inspired by a decorative ornament.[15] Other Islamic geometric problems may also have a hitherto unidentified historical context related to ornaments.

The craftsmen knew that the mathematicians had worked on some problems related to ornamentation and they regarded the solutions with respect, even though they probably did not understand the details and technicalities. The Persian treatise states [12, 185a] that the construction of a right-angled triangle such as EZT in Figure 8 "falls outside the *Elements* of Euclid" and requires the "science of conic sections". No drawing of a conic section occurs anywhere in the Persian treatise.

Of course we cannot exclude the possibility that a few mathematicians were also involved in the design and construction of geometric ornaments. The heptagonal pattern in Figure 6 is explained in our treatise in the language of the craftsmen, but since ^{c}Umar Khayyām lived in Isfahan at the time that the North Cupola was built, it is possible that he was somehow involved in the design. That a

[15] When Khayyam's text on the division of the quadrant was published in 1960 [13] and in 1981 [10], the modern editors had no way of knowing that the problem was inspired by ornaments. Around 1995 Özdural discovered the connection as a result of his study of the anonymous Persian treatise [15].

combination of mathematical learning and manual skill was possible in Islamic civilization is shown by the case of Abū Ḥāmid al-Khujandī (ca. 980), who was trained in Greek geometry and astronomy, authored a number of geometrical and astronomical works, and was also a superb metal-worker.[16]

The source materials that we have discussed in this paper give a fascinating glimpse into a design tradition about which little is known. Our knowledge is based to a large extent on one single Persian manuscript which is now preserved in Paris. It is likely that a systematic search in manuscript libraries in the Islamic world will produce many more relevant documents, and lead to a significant increase in our insight into the working methods of the medieval Islamic craftsmen.

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