Mathematics for the working Artist
Part II
An introduction to the qualitative theory of cones

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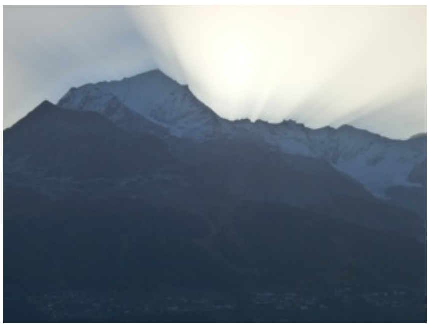
 

Abstract

The meaning of the term "cone" defined in this article is much broader and more flexible than the classical one. Our extension of this concept lays the foundations for a broad mathematical theory that could be used by artists. This article is illustrated by examples taken from mathematical and botanical sources.The powerpoint [S] is a kind of summary of this article.

1 Introduction

(From Wikipedia) ≪In a letter addressed to Émile Bernard dated 15 April 1904, Cézanne ambiguously writes: "Interpret nature in terms of the cylinder, the sphere, the cone; put everything in perspective, so that each side of an object, of a plane, recedes toward a central point."≫

Except during the Renaissance, painters have not studied and deepened the mathematics underlying their works. In the best cases, they have used what mathematicians have thought of and discovered.

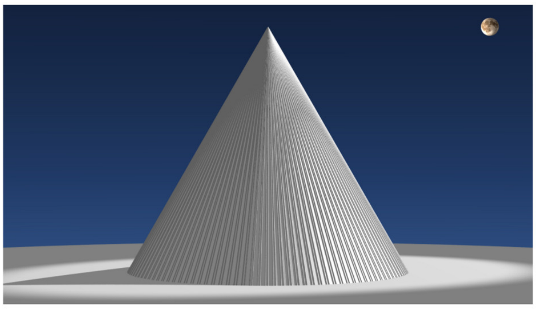
In particular, many mathematicians have developed the study and the representation of their objects using numbers as a powerful coding system. Geometers and topologists use a more direct and intrinsic approach to define and understand these objects. Knots, polyhedra, spheres and tori have been the main fundamental objects they looked at and used to that aim.

In this article, I would like to focus the attention on the cones mentioned by Cézanne, and to what can be done with these cones. In the past, with the work by Apollonius and his successors involved in the theory of conics and quadrics, cones have played an important role in geometry, then, much later, in mechanics and physics. Mathematicians did not emphasize the fact that cones are also present in perspective theory, thus, in some sense, in projective geometry : remind the ≪central point≫ Cézanne was evoking.

The notion of cone I define and use here is much larger and flexible than the classical one. The introduction of different manners to assemble these cones through identification and attachment along singular elements allows the construction of a much richer collection of objects than the one obtained by the use of Cézanne'tools.

The article, illustrated by examples borrowed from the mathematical and the vegetal worlds, does not address the mathematician who would like to develop and expand the mathematical content along several directions[1] (projections, apparent contours, duality, transformations, enumeration, algebraic and numerical representations, sections, trajectories, in Euclidean spaces or not). It addresses the artist who might wish to play with all these cones and create new beautiful works for the pleasure of our eyes and of our mind.

2 Singularities

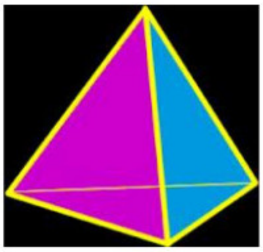


[1]The mathematical theory behind this paper is the enormous theory of fiberspaces with singularities, whose first chapter is the theory of cones.

A quasi standard cone, a view by Jos Leys
Figure 1

In a previous paper [1], several concepts and tools have been set forward, in particular the ones of singularity and of singular part of a shape.

Typical examples of singularities are for instance the vertices of a polygon in the plane, or the vertices of a polyhedron in the usual space, like the four vertices of tetrahedron, the six vertices of the octahedron (images from Wikipedia) :



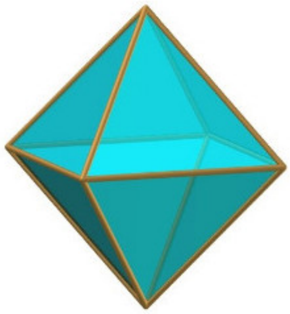


Figure 2

D being a local connected domain of the shape, we shall say that it is homogeneous of dimension n, if any neighborhood of any point of D has the *topological dimension n.*
For instance :
- any edge of the tetrahedron, without the two vertices which close that edge, is a 1-dimensional domain ;
- any face of the tetrahedron, without the triangle which borders it, is a 2 dimensional homogeneous domain.

Any subdomain of D whose topological dimension is $k < n$ is a *potential singular part* of D.
Thus, any point ($k = 0$) of a face of a polyhedron is a potential singular point, any curve ($k = 1$) drawn on the face, is a potential singular part.

Vertices of polygons and of polyhedra are not only potential singularities. They will be defined as *(incarnated) singular points.*

3 Cones

3.1 Introduction

Traditionally, there is nothing inside a tetrahedron : it is an *hollow object.* But it may be filled with matter : it becomes then an heavy die, a *full object.* We shall make the distinction between hollow cones and full cones.

In this paper we shall consider Euclidean spaces only. All the cones **C** we are going to consider will be here defined[2] by the three following ingredients, the two first ones a priori lying in an n-dimensional Euclidean space :

1) a *vertex* V (topological dimension 0), called the *main vertex* or the *apex* of the cone.

2) a *basis* denoted by $B(f)$, a closed domain of topological dimension $n-1$, or by $B(h)$, which is the boundary of $B(f)$, thus a closed domain of topological dimension $n-2$. [In the following example (Figure 3 left), $B(f)$ is supposed to be a full triangle].

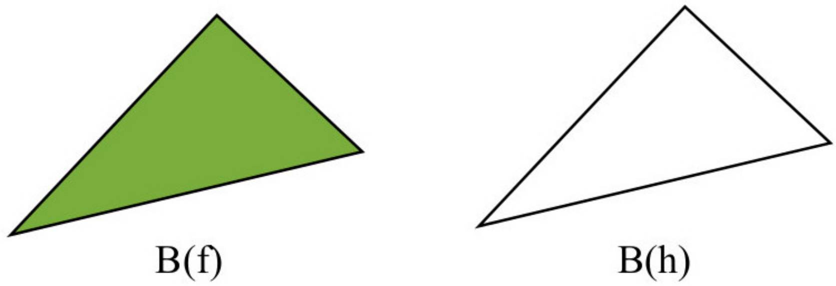


Figure 3

3) an interval $I \subset \mathbf{R}$ (topological dimension 1) called the *standard generator* or *fiber*, each of whose inclusions into the cone through a non decreasing differential mapping: $i_P : I \to \mathbf{R}^n$ is a curve that joins a point P of the basis to the apex V. This curve is called the *local fiber* at P.

Definitions: A cone $\mathbf{C}$ with $B(f)$ as a basis is called a *full cone of dimension* n. A cone with $B(h)$ as a basis is called an *hollow cone of dimension* $n-1$. We shall say that a *cone is linear* if all the local fibers are intervals (all the local inclusions are linear mappings).

3.2 Standard basic examples

3.2.1 As an example of full cone, we may choose the full tetrahedron. We can look at it as a (linear) cone if we:
- choose a vertex of this tetrahedron and name it V.
- consider the opposite closed face to V - its topological dimension is 2 - and name it $B(f)$.
- consider the intersection of the tetrahedron with any line which cuts $B(f)$ at any P and joins V. This intersection is the local inclusion of the interval I, the fiber at P.
The topological dimension of the full tetrahedron is 3 as being equivalent to a full sphere called a 3-dimensional ball. The boundary of this cone is the *complete* hollow cone associated with the full cone.

[2]This definition can be understood as the result of the Bourbakist point of view : Bourbakist in the good sense, i.e. a structuralist point of view, looking at the elements of an object which characterize its structure.

3.2.2 Now, from the full tetrahedron, we can extract an other hollow cone by considering the three faces adjacent to V :
- V remains the apex of that cone,
- its basis $B(h)$ is now the close curve, i.e. the hollow triangle which bounds $B(f)$, the opposite face to V,
- the intersection of the tetrahedron with any line which cuts $B(h)$ at any P and joins V. This intersection is the local inclusion of I.

3.2.3 Simpler, Figure 4 shows a green triangle which is a full 2-dimensional linear cone lying in the usual plane. Its basis bb' is the opposite side to the apex V. The boundary of bb' is the set of the two points b and b'. The two hollow 1-dimensional corresponding cones appear on the right.

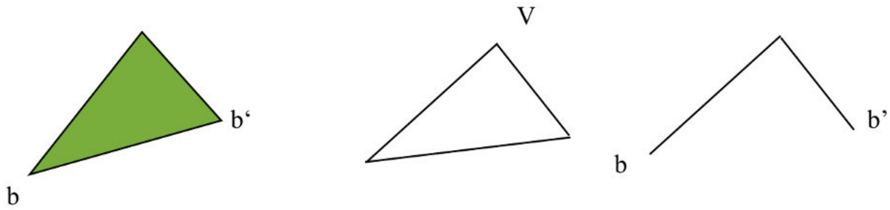


Figure 4

3.2.4 When $n = 2$ (the plane), Figure 5 shows the example of four hollow 1-dimensional cones whose vertex V is an antibubbling singularity (up) or a bubbling singularity (down). The basis here has two points which are not visualized here. The curves g and g' are local inclusions of the interval I. I shall call that cone a ≪Chinese hat≫. In each case, one sees two cones with the same apex V: the larger one is non linear, the linear one comes out from the previous one by considering the tangent lines in V to g and g' respectively.

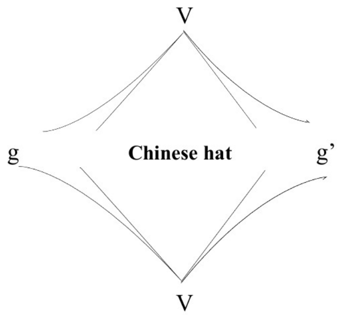


Figure 5

3.2.5 Consider any family of knots in the usual 3-space (simpler, a pencil of conics), points in that space which play the rôle of apices: look at the mountains you obtain !

4 A few remarks and definitions

4.1 Any n-dimensional full cone **C** with apex V and basis $B(f)$ gives birth to two (n-1)-dimensional hollow cones with apex V: the complete hollow cone of

C, which is the boundary of **C**, and $\mathbf{C}^c$ the *coat* of the full n-cone whose basis is the boundary of $B(f)$. This coat is included in the boundary of the full cone. Conversely, a hollow (n-1)-cone with basis $B(h)$ can be the coat of an infinity of n-cones. Any two such n-cones share the same boundary $B(h)$ of their respective basis $B(f)$ and $B(f)'$. They wear the same coat. These n-cones will be named the *wearers* of the (n-1)-cone.

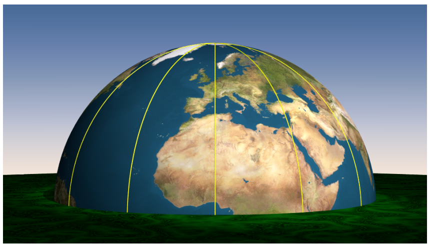
4.2 Figure 5 shows the example of a linear cone which is defined by the tangents at its vertex V to the fibers of a given cone, with the property that the angle between the tangents is not null, nor equal to π.

Cones with such a property, i.e. the tangent cone is not a linear (n-1) subspace, will be called *rough cones*.

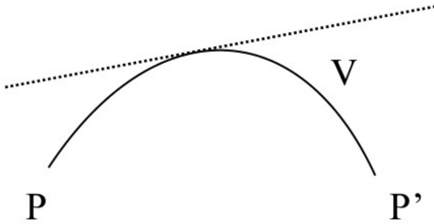
A rough cone has a *unique linear tangent cone*.

But conversely, a linear cone has an infinity of rough cones for which it is their common linear cone.

4.3 If the tangent cone is such a (n-1) linear subspace, the cone is a *soft* or *spherical cone*. Any point of an half-circle, of an half-sphere, is thus a *spherical apex* of a soft cone and a potential singularity. It becomes an incarnate singularity when its location becomes defined by the supplementary data of a directional line for instance.



Half-sphere as a soft 2-cone in the 3D-space: its basis is a circle A view by Jos Leys



Arc of circle as a soft cone : an edge joining its basis B(h) = {P,P'}

Figure 6

A particular interesting situation happens when the main vertex of a cone is located on its basis. In that case, we shall speak of a *basic cone*. Basic 1-cones play a fundamental role.

4.4 Let us consider the 1-dimensional hollow cone named the *cusp*[3] and defined by:
- the apex $V(0,0)$ is the origin of a usual orthogonal coordinate system of the real plane,
- the basis $B(h)$ of this cone is the set of the two points $P(1,1)$ and $P'(-1,1)$,

[3]The cusp is the most basic singularity. It has been used as a geometrical support in a study of the universal phenomenon of ambiguity.

- I is the interval $[0, 1]$, and the local inclusions of I in P and P' respectively are defined by the parametric equations:

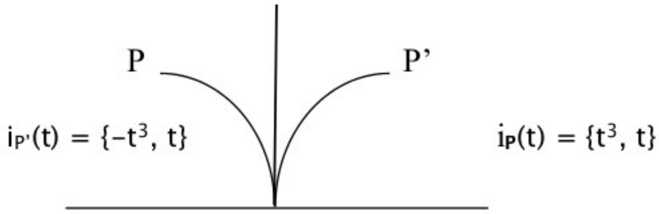


Figure 7

The line which joins the apex V to any point $Q(x, y)$ of the fiber $i_P(I)$ has a slope defined by the ratio $y/x = t/t^3 = 1/t^2$. When t tends towards 0, this slope becomes infinite, so that the tangent in V to this fiber is the vertical line. For a similar reason, the tangent in V to the fiber at P' is also the vertical line. In other words, that cone has a unique vertical tangent in V: the linear tangent cone is an half linear space.

A cone whose linear tangent cone is so degenerated will be called a *penetrating cone* or a *spine*.

4.5 Let us go back to the examples illustrated by Figures 5 and 7. In Figure 5, the upper cones seems to be the symmetric of the under cones with respect of the horizontal line. More generally, any cone has a *symmetric* one with respect to any domain parallel to the domain containing its basis.

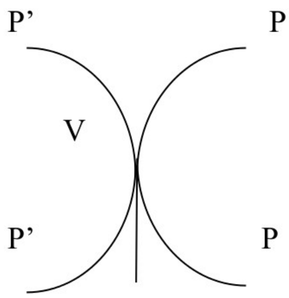


Figure 8

4.6 Let **C** be a given full n-dimensional cone with vertex V. Let B be a n-dimensional ball whose center is V: the boundary of that ball is the (n-1)-sphere centered in V. We suppose that the ball is small enough so that the common part to the ball and the cone is entirely contained in the cone.

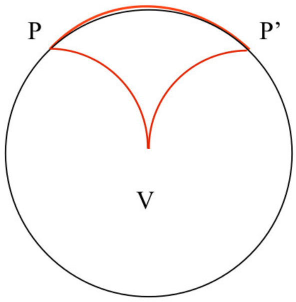


Figure 9

The *complement* **C*** of **C** in B is a full n-cone with the same coat as **C**.

Let us consider the ≪half≫ n-spaces through V. If **C** is contained in one such half-space, **C** will be named the *male part* of B, and called a *male cone*. Its complement is the *female part* of B and is a *female cone*.

4.7 Let $\mathcal{F}$ be a given continuous family of (n-1)-cones $\mathbf{C}_t$ parametrized by t belonging to I, with apex V_t and with the same basis B. Let $\mathcal{A}$ be the curve $t \to \mathcal{A}(t) = V_t$: this curve will be called *an axis* of the family.

Figure 10

Let T be a (n-1)-dimensional domain which is transverse to the axis, and σ_t be the section of the cone $\mathbf{C}_t$ by T. We suppose that for any $t' < t$ the closure of σ_t' is contained in the closure of σ_t.
Then the closure of $\mathcal{F}$ is a n-dimensional cone *foliated* by the cones $\mathbf{C}_t$.

Any (n-1)-cone **C** is the coat of an associated canonical wearer $\mathcal{F}(\mathbf{C})$. An axis of $\mathcal{F}(\mathbf{C})$ be also called an *axis* of **C**.

Discrete foliations of cones can be worked out in the same spirit.

4.8 Let **C** be a 1-dimensional cone, P and P' the two distinct points of its basis. Let us suppose that the curvatures at any point of the fibers are not null or infinite except maybe in V.
Such a cone, like the left one, might be named a *half smiling cone* if these curvatures have the same sign.

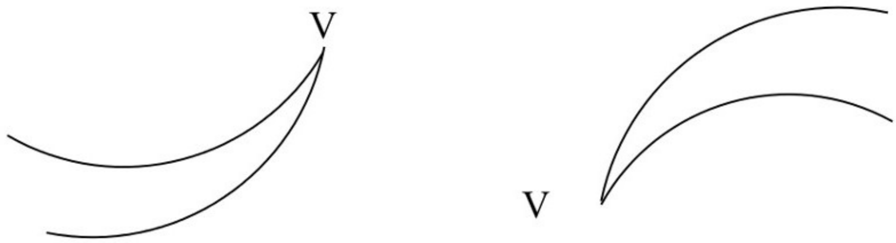


Figure 11

4.9 Let $\mathbf{C}_i$ be an (n-1)-dimensional cone embedded in a n-space and called the *motive*, $h(\mathbf{R}) = \Lambda$ be a curve of such a space (more generally a $k < (n-2)$ dimensional domain), and V a point of Λ. Let S_i the shape defined by $S_i = \Lambda \times \mathbf{C}_i$ so that V is the apex of a cone $\mathbf{C}_i$.
The shape S_i will be called a *regular conical excresence* of $\mathbf{C}_i$ along Λ, Λ being its *singular curve* or again its *handle*. Note that several S_i can share the same singular line, so that the union $S = \bigcup S_i$ of these local shapes can be taken into consideration.

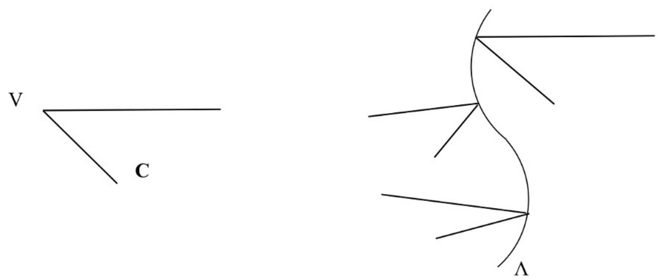


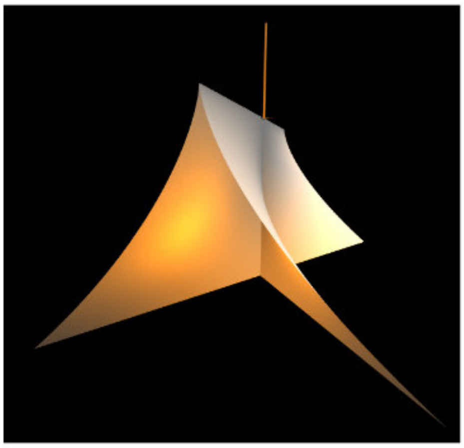
Figure 12

(More generally, we may suppose that, for each V, the corresponding cone is subjected to an eventually continuous controlled change of metrical properties).

Given a curve Λ in a n-dimensional space, a point V of that curve for which the tangent to the curve is well defined, a transversal subspace to the curve in V is a (n-1)-dimensional subspace which does not contain the tangent.

Conversely, suppose that V belongs to a shape S so that any transversal subpsace to V defines a cone $\mathbf{C}_v$ on S whose main vertex V is on Λ, then Λ is defined as a *singular curve* of S. When Λ lies on a cone, S will be named a *flag*. When the cones are full cones, we shall say that S is a *mountain* and Λ its *line of summits*.

A fairly nice mathematical example of the coat of such a mountain is the Whitney umbrella where Λ is a line :



The Whitney umbrella from www.algebraicsurface.net
Figure 13

4.10 Indeed, the way according to which cones are attached to singular domains is not restricted to the consideration of their main vertices. Any other singular part of dimension $k' < k$ of a (n-1)-cone, where $k < n - 2$ is the dimension of a domain Λ, can be attached to Λ.

The ≪leaf≫ of a fen is called a penna, which is made of pinnulae, here viewed as 2-cones, whose basis are attached to the singular curve named the rachis.
Figure 14

4.11 Let $\Gamma(\Delta)$ the group of symmetries of a part Δ of the basis of a cone $\mathbf{C}$. Δ induces the part $\mathbf{C}/\Delta$ of the cone, and $\Gamma(\Delta)$ will be called the *symmetry group* of that part $\mathbf{C}/\Delta$.

5 Compositions of cones

5.1 1-dimensional cones

5.1.1 Introduction

Let us first give a list of non spherical 1-dimensional hollow rough cones in a flat 2-dimensional space. Each one has tow edges : one of them will be called the *arm*, the other one the *anti-arm*.

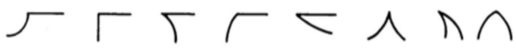
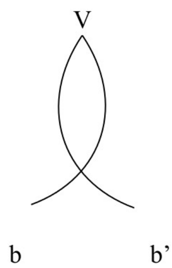


Figure 15

(It is amazing to compare this list published in 1976 [2], with the following by Dürer around 1528 [3]: 1976 - 1528 = 446)

Figure 15 bis

Let us add to that list a soft 1-dimensional hollow cone, like an half circle or an edge, a spine like the cusp, and the basic 1-cone, an edge whose one vertex is the main vertex of the cone.
Each cone of the list gives rises to a series of n-folded arms cones like this elementary one :



1-Folded arms

Figure 16

The basis of each of the cones $\mathbf{C}_i$ of the original list is a set of two points: $\{P_{i1}\ ,\ P'_{i1}\}$. Each such cone gives birth to an infinity of wearers (full cones) which can also be taken into consideration.

The boundary of a full 2-cone has: three points, the apex V and the two vertices of its basis, the two curves of the hollow cone that join the vertices of the basis to V (the arm and the anti-arm), and the curve of its basis $B(f)$ which joins the two vertices of its basis. All these curves may be viewed as singular elements of the full cone.

More generally, a fiber Σ of the cone is *singular* if it contains an element of curve Λ such that the intersection of its neighborhood with the cone is a flag S whose local conic motive is rough or a spine.

We are going to proceed to attachments of 1-cones along these different singular elements through processes of identification.

5.1.2 Self-attachment

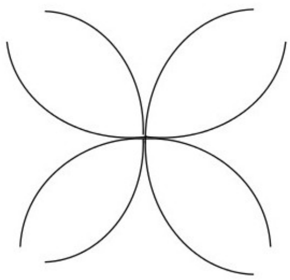
Given a 1-cone, the identification of the two points of its basis gives rise to a topological 1-sphere like a circle, while the identification of the two edges adjacent to the main vertex gives rise to a basic 1-cone.

Figure 17

5.1.3 Attachment of cones by identification of their apex

The attachment by identification of the apex of two basic 1-cones gives birth to one of the previous 1-cones.

Cones attached to each other by identification of all their apex to one of them will be called *spiders*. In that case, each cone of the spider could be called an *arm* or a *tentacle*. Figures 8 and 18 show examples of particular spiders.



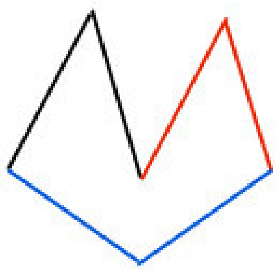
Spider or Flower or Bouquet
Figure 18

5.1.4 Attachment by identification of a unique point of their basis

5.1.4.1 Let Σ be a sequence of N various cones of the list, any cone $\mathbf{C}_i$ appearing n_i times in the sequence. Then two consecutive cones $\mathbf{C}_i$ and $\mathbf{C}_j$ - where j can be equal to i - are attached by a unique point of their basis, if only one point of the basis of $\mathbf{C}_i$ is identified with one point of the basis of $\mathbf{C}_j$.

In that way, we shall say that we have got a *garland* or *frieze* of 1-cones, or a flag if all the cones except one of them called the *handle* are attached to this handle.

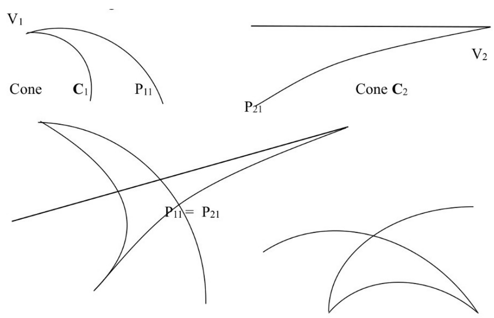
If the first cone of the sequence is attached to the last cone of that sequence, we shall say that the garland is *knotted* or *polygonal* : we can understand a knotted garland as as a knot with singularities.



$\mathbf{C}_1$ is attached to $\mathbf{C}_2$ which is attached to $\mathbf{C}_3$ which is attached to $\mathbf{C}_1$
Figure 19

A polygonal garland with $2N$ edges can be constructed with N cones.
Polygonal curves with an odd number of edges $2n+1$ may need n rough or penetrating cones plus a soft cone. But an other way to get such a polygonal garland is to divided each edge into two attached parts, and then to use $2n+1$ cones to get it.
Each knotted garland generates a spider, its dual, but the converse is not always true. Except lines, any other curve in any n-dimensional space can be decomposed in such rough or penetrating hollow 1-cones, so is a garland of hollow 1-cones.

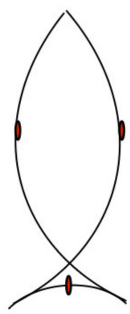
5.1.4.2 Here is an example arising from the mathematical butterfly in catastrophe theory. The following local section of this surface can be viewed as a garland of the two following cones :



The Bird, the Swallow Tail
Figure 20

Appropriate deformations of the above drawing give birth to a stylization of a bird.

The following shows a stylized fish as, first, the visualization of a white 1-dimensional full cone where all the fibers have a unique other common point than the vertex - but of course they could have many such common points.



The fish
Figure 21

But if we introduce fictive or virtual vertices in the middle of each edge (the red points on the figure), we then define three hollow 1-cones with main vertex respectively V, b and b' from which the fish can be reconstructed.

5.1.4.3 Suppose a given 1-cone imbedded in a n-dimensional space. The possibilities to attach an other 1-cone to one point of the basis of the given cone is infinite, being ruled by the group of rotation of that n-space. Given constraints can of course reduce this set of potential possibilities.

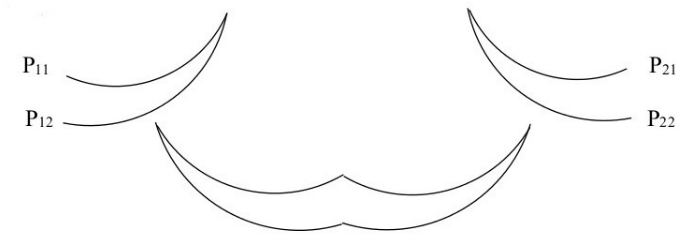
5.1.5 Attachment by identification of the two singular points of their basis

Base of 1-cones are very elementary. Given a process of attachment (the choice of the manner to identify the basis), there are infinite possibilities of attachment of cones to a given one imbedded in an n-space, each possibility being defined here by an element of the group of rotation of the (n-1)-space.

We shall call a p-*flag* the set of $p-1$ 1-cones so attached to a given 1-cone, the handle.

Here is an easy example in the plane (n = 2).

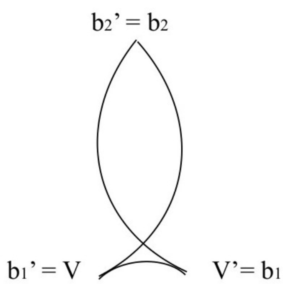
The given 1-cone is $\mathbf{C}_1$, while the 1-cone to be attached to it is $\mathbf{C}_2$, indeed a clone to the first one:



Smile and Moustache
Figure 22

There are only two ways to attach the two cones with the same identification of their basis. The first one gives a perfect superposition of the two cones since they have the same shape (identity of $O(1)$). The second way, a symmetry, gives rise to a true smile; or a moustache !

Here is an other way to construct the fish where a point of the basis of a first cone is the apex of a second cone.



The contour of a fish built from two symmetric 1-cones
(can also be the complete hollow 1-cone of a fish)
Figure 23

5.1.6 Attachment along singular curves

We have been considering attachments along the apex V and the elements of the basis. We now consider identification of the singular curves joining V to the vertices of the basis, two such curves being able to be identified if and only their curvature is the same.

Given two red 1-cones with apices V_1 and V_2, this identification first implies that the identification of V_1 with V_2, and the identification of a vertex b_1 of the basis B_1 with a vertex b_2 of the basis B_2 : in other words, the attachment 4.1.2 and 4.1.3 have be done simultaneously, but that is here a part of the process since the identification concerns all the points of the singular curves.

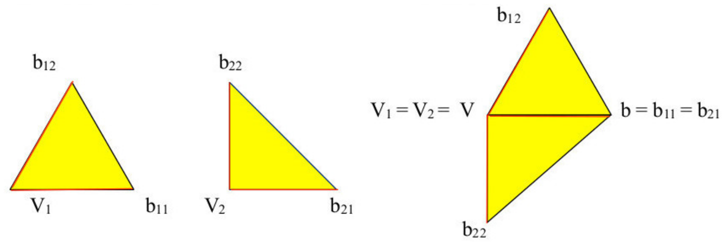


Figure 24

A sequence of N 1-cones in an n-space ($n > 2$) which are attached along a singular curve g of a given one will be called a *N flag along g*.

The use of less usual 1-cones gives birth to unusual shapes, especially if all the processes of attachement are used all together.

5.2 2-dimensional cones

5.2.1 Introduction and examples

5.2.1.1 First, let us show a very few 2-cones - the mathematical images are borrowed from the net, see for instance ≪images of algebraic surfaces≫:

The same operations of identification and attachment can be worked with n-dimensional cones. Here are a few classical pictures of assemblies of 2-dimensional cones attached along singular parts of their boundary, apices, edges, basis:

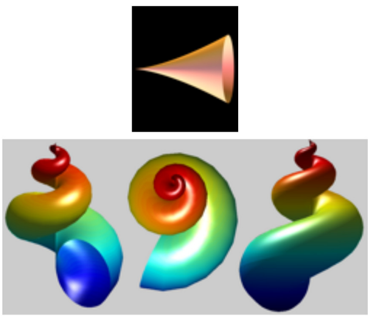
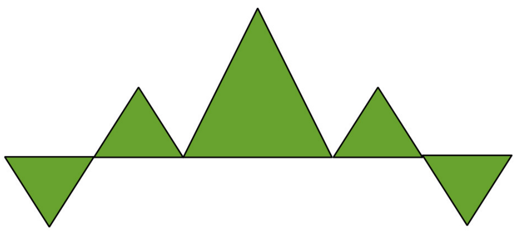
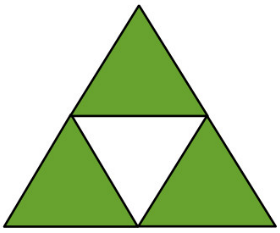


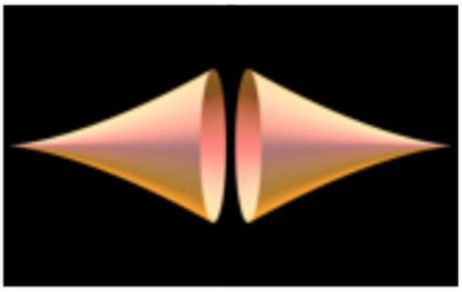
Figure 25

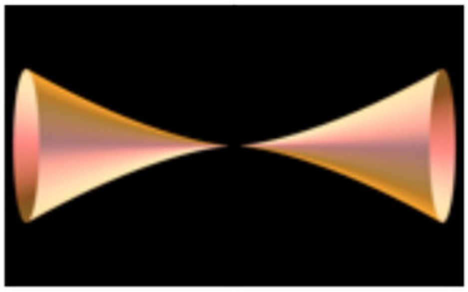


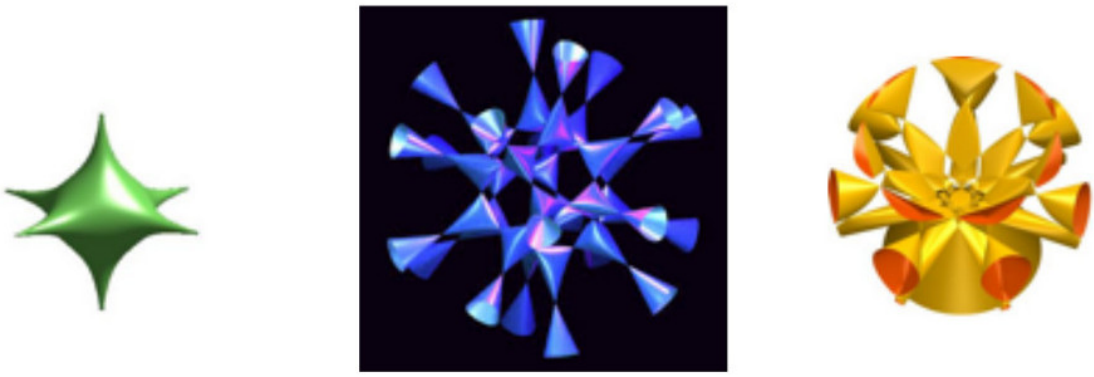
A garland (or frieze)



Serpinski motive as knotted garland







Classical nodal surfaces
Figure 26

It is easy to extend these examples by adding more cones of different sizes, or to start with other polyhedra including Gosset polyedra, using apices defined through discrete subgroups of $O(n)$, and reproduce similar constructions of attached cones.

5.2.1.2 The vegetal world is also a source of examples. Let us first consider the following standard mathematical 2-cone and one of its incarnation as a leaf of the lily of the valley in the usual 3-space:

Figure 27

This incarnation has the nice property to show possible fibers of the cone. Nature is now going to attach along their basis two clones of that cone. Here they are:

Figure 28

5.2.1.3 Let now us consider the following mathematical smiling 2-cones and the two following leaves :

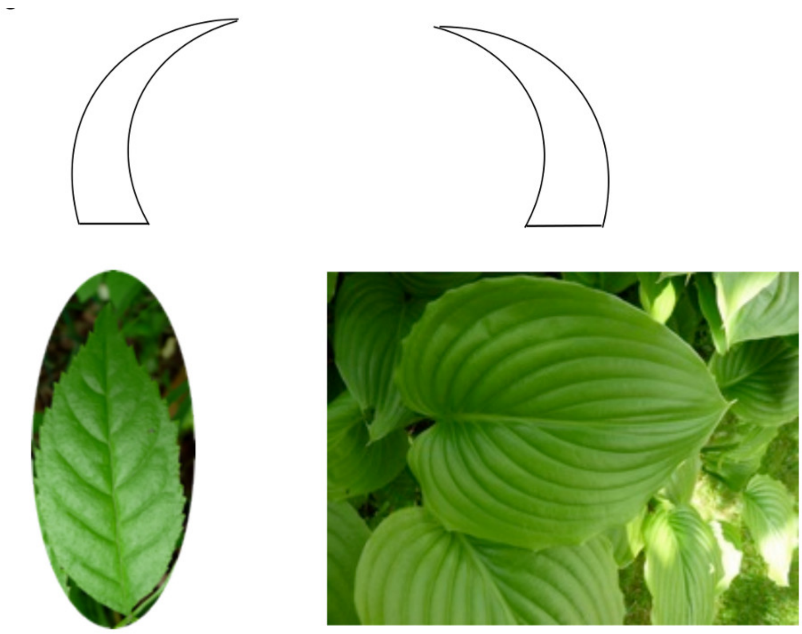


Figure 29

The left leaf shows two similar sequences of half smiling 2-cones, more or less symmetrically located on a singular curve like in Figure 14. But each central cone is attached along a singular line of its border to two another cones, one above and the other under itself. On the leaf of the right, moreover, all the apices meet at the top of the leaf, on the singular curve. Indeed, these cones getting very thin give rise to the ≪fibers≫ which appear on Figures 22 and 23.

5.2.1.4 Let us know consider the following leaf :

Figure 30

We discover that the so-called previous generic 2-cones which seemed to appear on Figure 30 say are indeed mountains in the sense we used to caracterize the Whitney umbrella (Figure 13).

5.2.1.5 Let us give here a few simple other examples of 2-dimensional objects created with more simple 2-cones using the standard techniques of attachment:

For instance, the 2-cones of Figure 25 can be created by the classical identification of the two edges adjacent to the apex of convenient ≪triangular≫ 2-cones : cutting and opening the given 2-cones along a curve through their apex give rise to the convenient triangular 2-cones.

The standard 2-band in the usual space can be created by attachment of two triangular 2-cones **C** and **C'** like full half-smiles, but which can have any specific shape:

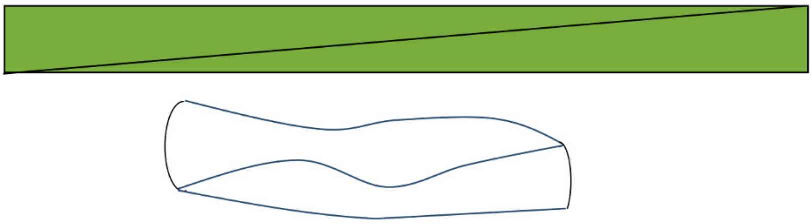
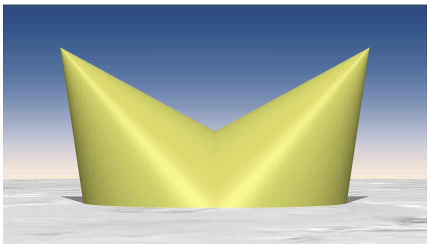
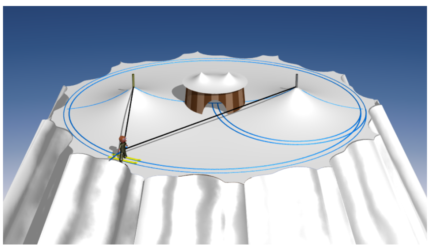


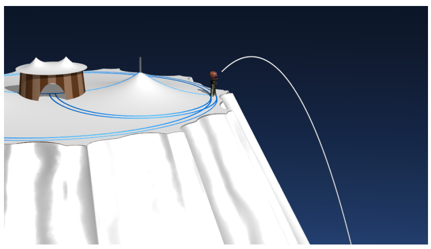
Figure 31

Twist the band as you wish in the usual 3-space, attach the corresponding opposite sides and get Möbius bands, deformed cylinders and tori.

5.2.1.6 There are many ways to assemble cones of different shapes and to create landscapes.
Here are two examples of such constructions: the first one, among the simplest, show a double cone arising from the identification of the basis of two linear cones in the usual space, the second one was made by nature, a few years ago.







Double cones by Jos Leys

Figure 32

5.2.1.7 Here is a final remark about 2-cones, one that more generally applies to n-cones. In the usual 3-dimensional space, let **C** be a hollow 2-cone with basis $B(h)$, D a 2-dimensional linear subspace which meets that cone. The common part of **C** and D is a plane curve β. This curve may have singularities and multiple common points. Then the part of the fibers through these points between D and the apex V of the cone are singular curves of the cone.

5.2.2 Creation of 2-cones from 1-cones

We are now going to look at two main techniques to create 2-cones from 1-cones.

5.2.2.1 From full 1-cones, by attachment:
Let us first recall that a full 1-cone is indeed a 2-cone since it is a 2-dimensional surface.

The hollow tetrahedron gives an example of the attachment along singular lines of a sequence of 3 standard linear full 1-cones:

$$(V_1, b_{11}, b_{12}), \qquad (V_2, b_{21}, b_{22}), \qquad (V_3, b_{31}, b_{32})$$

attached one to the other through the identifications of the singular lines

$$(V_1, b_{12})\texttt{@}(V_2, b_{21}) \qquad (V_2, b_{22})\texttt{@}(V_3, b_{31}) \qquad (V_3, b_{32})\texttt{@}(V_1, b_{11})$$

More generally, we shall call a polyhedral 2-cone such a 2-cone constructed from a sequence of full 1-cones with a cyclic presentation of their singular lines. Note that generically, this kind of 2-cone is not a standard polyhedron nor a part of such a polyhedron.

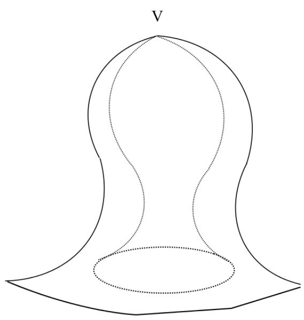
Flags of 2-cones can be constructed by attaching other 2-cones along a singular line of one of them, or along the basis of one of them, or along a curve of excresence.

5.2.2.2 From hollow 1-cones, by local transformations:
1) Let **C** be a (n-2)-cone in an n-dimensional space, A be a curve which contains the apex V. We denote by A_C the set of points Q of A for which L_Q

the linear orthogonal (n-1)-dimensional subspace to A in Q meets $\mathbf{C}$. Denote by $\mathbf{C}(L_Q)$ the intersection of $\mathbf{C}$ and L_Q.
Let ρQ be a continuous translation and/or a continuous rotation of $\mathbf{C}(L_Q)$ around Q in the subpsace L_Q giving birth to the trace $T\mathbf{C}(L_Q)$ of $\mathbf{C}(L_Q)$ in that subspace. We suppose that ρQ is a continuous function of Q. From the geometric point of view, we can also suppose that each local transformation changes the local sizes.

The union of these traces $T\mathbf{C}(L_Q)$ when Q moves continuously on A_C is a (n-1)-cone.

Here is a trivial example where ρQ is a 360° rotation, A is a vertical line. Starting with an half smiling cone, we may get for instance the following hollow 2-cone. We might call the corresponding full 2-cone the ≪bell≫, or the ≪hat≫.



The bell
Figure 33

2) More generally, A does not contain V. Then we do not get a cone in general, but the coat of a moutain.

A fairly simple example is the Whitney umbrella that can be obtained by translating a Chinese hat, without any metrical transformation of its size.

From the metrical (geometrical) point of view, the presence of local symmetries of the basis is of some interest. One can impose in particular that the vectorfield which acts on the transversal sections $\mathbf{C}(LQ)$ keeps on the associated group of symmetries. Then we get a priviledged axis.

5.2.3 Full 2-cones, the 3-ball and the 2-sphere

Let us consider a family of full linear 2-cones $\mathbf{C}$(t) like full triangles. From the topological point of view, one can represent them by 2-cones whose basis are arcs of circles.
Let $A(t)$ the area of the cone $\mathbf{C}$(t). We suppose that the mapping t $\rightarrow$ A(t) where t describes the interval [0, 1] is continuous, with A(0) = 0, and A(1) = A.
Now, in the usual 3-space, let Λ a vertical interval, and S the shape, the flag defined by a regular conical excresence of the family of $\mathbf{C}$(t) along the handle Λ.

Here is (left) a vegetal example of such a shape showing $\mathbf{C}(1)$ and Λ, together with its symmetric (right):

flags with homothetic cones
Figure 34

Let $1(t)$ and $2(t)$ be the arms of $\mathbf{C}(\mathrm{t})$ and call the sets $F(i) = \{\bigcup\{i(t) \mid t \in [0, 1]\}$ the i-face of S, where $i = 1$ and 2.

Consider n clones of S, S_1, S_2, ..., S_n, and their respective faces $F_k(i)$ where $F_k(i)$ is the i-face of the clone S_k.

Identify their handle to get a flag, then identify the face $F_k(2)$ with the face $F_{k+1}(1)$ for $k < n - 1$, the face $F_n(2)$ with the face $F_1(1)$.

Topologically, the result is a 3-ball whose boundary is the 2-sphere : one may taste an equivalent final following conclusion.

Figure 35

6 Singularities again

6.1 Creation

The pinching process [1] is a standard process to create singular sub-domains. The creation of a singular point can be practically worked out in the following way. Choose the location in the object close to which the singular point should appear. Consider a small ball containing this location and a point V inside the ball but out of the object. The intersection of the ball with the object will be the basis of a hollow cone with apex V such that the object and the cone share the same tangent space along the basis. Attach the cone to the object and cut off the interior of the basis.

Note that when the object is locally convex, the resulting singularity V can be bubbling or anti-bubbling according to its position with respect to the object.

A physical equivalent way to create a singularity consists in choosing a point V on the object and to draw out the object along curve through V. Such a process has been for instance used by Philippe Charbonneau to create the following sculpture:

Biconique 2 by Philippe Charbonneau
Figure 36

Here, the object is a curve, the knot called the trefoil which bounds a Möbius band. The curve was drawn out at two points V and V' which have been fixed up on a vertical rigid axis.

More complex sculptures could be similarly worked out with any other regular torus knot.

6.2 Suppression

6.2.1 The first natural process is to smooth the object by suppressing locally the cone and substituing to it a small half ball or half sphere. We may call this process the rounding of the singularity.

I shall show a very few reasonably good home made photos first for the pleasure of the eyes.
The first group of photos illustrate the internal symmetry of flowers and the layout of their petals viewed as cones. Indeed, it seems to me that the main symmetries of the floral world are of order : 2, $2+2$, 4, 3, $3+2$, 5. $2+2$ means a superposition of orthogonal symmetries of ordre 2. Similarly, $2+3$ means a superposition of a symmetry of order 2 and a symmetry of order 3. Frequently, the order of these fundamental symmetries is multiplied by an even number.

Figure 37

The second group of photos illustrates the rounding of the singular parts of some polyhedra which appear as buds of flowers.

Here, it is interesting to notice that the visible part of the complete flower itself (right) has the shape of a half octahedron.

The bud of a poppy and its flower
Figure 38

The bud of a peony

Figure 39

6.2.2 Paragraph 4.7 introduced a notion of foliation of a cone. This notion does not fit exactly what can be observed in nature. A better approach consists in introducing a notion of multiple protecting covering - richer than the notion of (simple) covering commonly used in mathematics.

For instance, let us consider the full 2-cone we have met in 3.2.3 (Figure 41, left), and its boundary, its associated complete hollow 1-cone, represented by the red triangle (Figure 40, middle). It is viewed as a simple covering of the full triangle. Since it has no thickness, we may cover the full triangle by any number n of replica of the hollow cone: they constitute a *multiple covering* of the full cone.

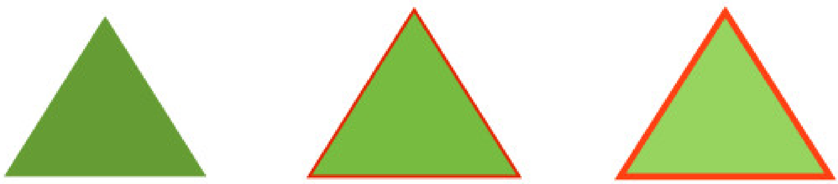


Figure 40

Consider now a tetrahedron as a cone $\mathbf{C}$ with apex V, whose basis is the previous full triangle. Its coat C^c is a hollow 2-cone whose basis is the red triangle of Figure 42. Consider now an other hollow 2-cone with the same apex V, but whose basis is the blue triangle.
The singular points of the red basis of the given 2-cone are contained in the regular part of the blue basis of the second 2-cone. We shall call that second cone a *protecting covering* of the first one.

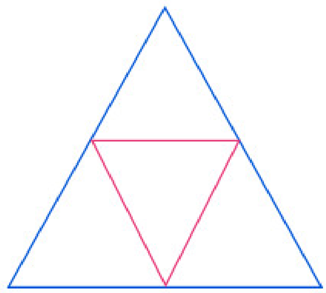


Figure 41

Indeed the second cone is ≪protecting≫ the singular lines of the first cone.

If you iterate the process of protection of the successive 2-cones, together with a rounding of the whole construction, you get something similar to the bud of a flower caracterized by an appropriate foliation.

As an example, we may choose the bud of a rose - the rose might have a $3+2$ symmetry.

Figure 42

7 Exfoliations

In order to create new shapes, we have intensively been using attachments along singular parts. If we think in physical terms, giving some thickness to a 1, 2 ... n-dimensional domain, the k-one will be understood as less strong than the $(k+p)$ one. Thus a singular part of an object belongs in some sense to the weak part, to the most fragile part of an object.
Then the attachment of two objects along some of their singular part may show some weakness, especially if the quality of the glue or of the soldering is not the best.
That is a reason which encourages the creation of protecting coverings.

We shall call exfoliation the inverse process of creation. As it is working in the floral universe, it consists in disconnecting an object along its singular parts, through local processes of separation, of detachment.

From the metrical and physical point of view, the process of attachment is not brutal in general, but is progressive, and can be numerically controlled in time according to the point of the singular part which is reached. The operation of exfoliation has similar properties, but can be run faster than the one of creation.

Since an apex is a 0-dimensional domain, exfoliation generically begins with such singular points. If we imagine the presence of a multicoloured cloud of 1-cones, exfoliation, a big-bang coming with the vanishing of the apices gives rise to an other cloud of arms and anti-arms.

Exfoliations of polyhedra give rise to many new beautiful flowers.

8 Conclusion

The topological theory which has been presented here is fairly simple, even perhaps naive. But giving also rise to a large amount of mathematical questions, its fecondity is rather a proof of its interest. In higher dimensions, our usual mathematical tools are unable to classify singularities. We may hope that the topological approach will permit us to go further. In other respects, the construction of an algebraic topology based on cones is more complex than the classical one, but the fact that a non linear triangle remains the assembly of three 1-cones, that several ways to attach cones can be used, suggests that a finer and a richer theory could be developed. It is worth noticing that a classification of cones seems to be impossible since it includes the classification of the basis of cones, which can be cones themsleves. That is why I have choosen, after the title of this article, to symbolize this theory of cones by the drawing of the snake which bites its tail.

From a pedagogical point of view, the theory is very pleasant: it is accessible to everybody, permitting the creation of a multitude of 2D and 3D cones, shapes and compositions, using modelling clay, strings, scissors, paper, pieces of cardboard, glue, and a brush. Later, software permitting, we may be able to make these constructions on computers. Using the set of these tools, an imaginative artist could have already created all the objects that have been shown on Figure 26 for example.
Via the concepts on which it stands, via the creations it allows, the theory stands in some sense at the junction of mathematics and art. Through the constructions he imagines and shapes, born of his hands, the child, the budding artist will express his dreams, and perhaps will reveal talents which will one day be expressed in an artistic activity, one of the most original of man, whether engraved in matter, or simpler and purer worked by the mind.

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