PICTURES OF THE PROJECTIVE PLANE

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Dedicated to Günter Törner on the occasion of his 60th birthday

Günter Törner started his mathematical career with a doctoral dissertation on *Hjelmslev planes.* These planes were named after the Danish mathematician Johannes Hjelmslev (1873 – 1950), who defined them as a sort of „natural geometries" meaning that distinct lines may meet in more than one point, like it may happen in a real drawing. There is a natural homomorphism $H \rightarrow P$ from a Hjelmslev-plane H onto an ordinary projective plane. The geometric axioms for H are in the "desarguesian case" such that H may be co-ordinatized by a ring that has a unique chain of ideals. Toener went on to study these rings in detail, which today is the main topic of his mathematical research beside his activities in mathematical education and his service to the Deutsche Mathematiker Vereinigung. On top of all this he is interested in the relations of mathematics and art and has even organized meetings on this topic. That is why I have decided to offer him some remarks on the pictorial history of the ordinary (real) projective plane P. I am sorry that I do not know of any nice pictures of Hjelmslev planes or their uniserial rings of coordinates.

Pictures of the projective plane

I will just line out the historical steps towards pictorial representations of the projective plane, providing historical pictures whenever possible. Unless stated otherwise, drawings are by the author. Ultimately the projective plane stems from the study of perspective in the Italian Renaissance. In our terms, this means to direct ones attention to the lines of sight emanating from the eye of the painter. The lines of sight may be cut with a sphere surrounding the eye and each line gives rise to the identification of two antipodal points of the sphere.

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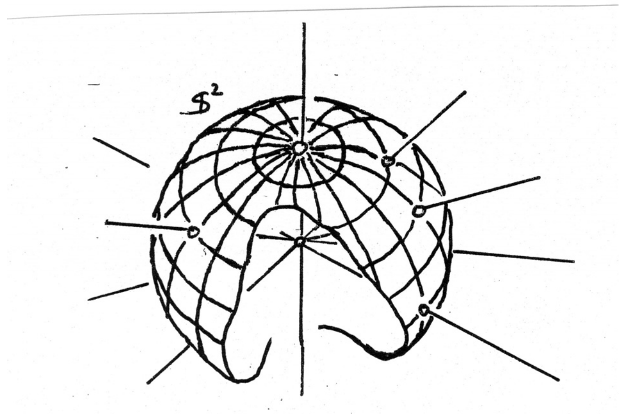

Fig. 1: Sphere with lines of sight

1. Jakob Steiner

In the first half of the nineteenth century, projective geometry came to the foreground of mathematical interests. One of the protagonists of synthetic projective geometry was the swiss mathematician Jakob Steiner (1796 – 1863), who found the first picture of the projective plane. It is what we today see as an embedding into 3-space (or maybe better into 4-space). He defined the mapping of the two-sphere S^2 into R^3 by

$$(x,y,z) \rightarrow (yz, xz, xy).$$

Since he had found this surface during a stay in Rome, Steiner called it "my Roman surface", but did not publish it. He told his friends Weierstrass and Kummer about it, who published it posthumously. In the year of Steiner's death, Kummer presented the first plaster model. Because the mapping identifies antipodal points of the sphere, the model gives us a picture of the projective plane. Steiner himself studied it in the connection with conic sections.

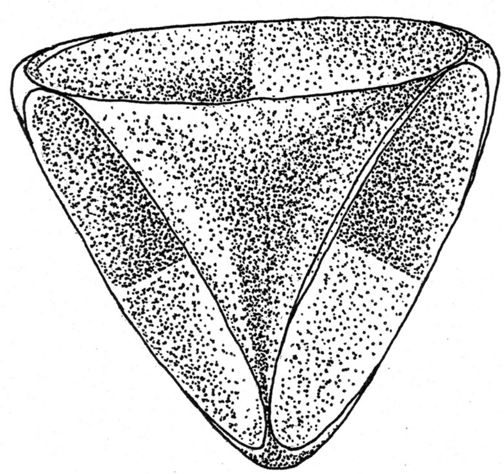

Fig. 2: Steiner's Roman Surface[2]

[2] For historical and modern details see:
Steiner Werke II, Nr. 32, Kummer 1863, Hilbert an Cohn Vossen 1932 Appendix p. 300/301, for excellent pictures of a plaster model Fischer 1986 I and for more details Pinkall in Fischer 1986 II, 108/109.

2. August Moebius

The second mathematician who found a picture of *P* in quite a different context was August F. Moebius (1790 – 1868). In 1858-65, he was looking for a way to determine the volume of a polyhedron from its (triangulated) surface. Before him, people had found ways to speak meaningful of the area of a closed plane polygon. Moebius realized that here was no way to generalize this to polyhedra. From the local aspects of the surface one can not tell if it contains a volume at all. His famous "Moebius strip" could be part of the surface and hence there was no way of speaking of an inside or outside of the surface (Moebius p. 482 – 485).

The minimal example presented by Moebius goes like this: First he defines his strip by a sequence of 5 triangles

ABC, BCD, CDE, DEA, EAB,

to be identified along corresponding sides.

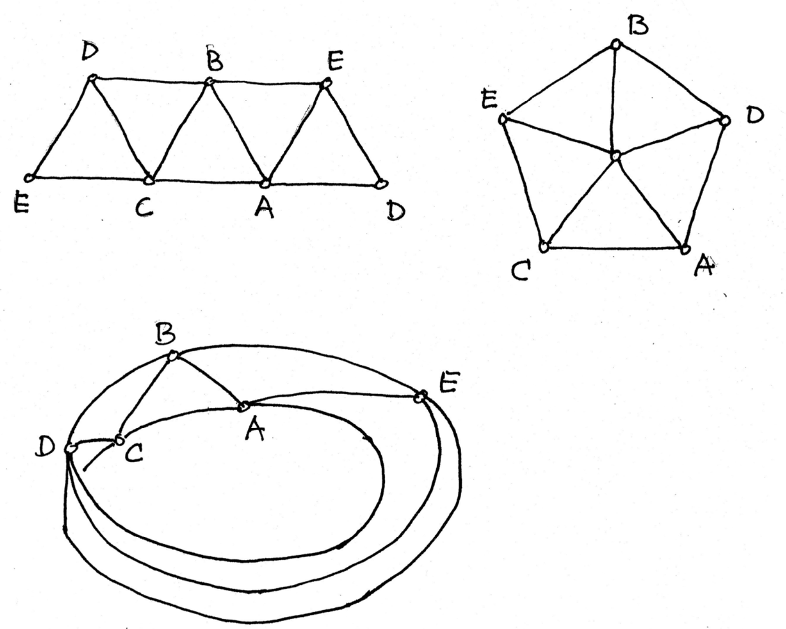

Fig. 3: Moebius` strip and lid

The strip is not a closed surface, so he takes a point F (outside the strip) and defines a "lid" by 5 more triangles

FAC, FBD, FCE, FDA, FEB,

to be glued to the corresponding sides of the strip. Now there is no free edge left and the result is a closed (non-orientable) surface *M.* Here Moebius did not provide a picture. We, however, may try and see the self-intersection of *M* along the line FX in Fig. 4. Interested readers may try for themselves to put this triangulation on a picture of a cross-cap projective plane as defined in the next section. If we count the faces f, the edges e, and the vertices v of *M*, we get the Euler characteristics

$$f - e + v = 10 - 15 + 6 = 1$$

of the projective plane.

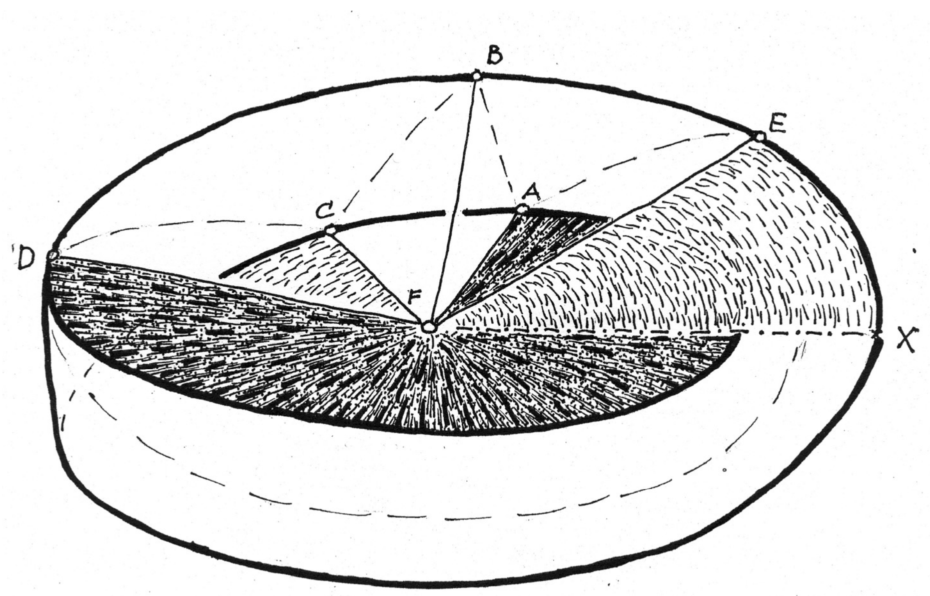

Fig. 4: The projective plane after Moebius, the self intersecting triangles FAC and DFE are shaded.

3. Interlude

Before going on we will have a modern look back at Moebius`construction and the ordinary topological representation of *P* as a (sphere with) cross cap. What surface results from identifying antipodal points of the sphere *S*? Instead of identifying, we may select one of the two antipodal point and delete the other one. Proceeding thus, we split the sphere into two polar lids and a central belt.

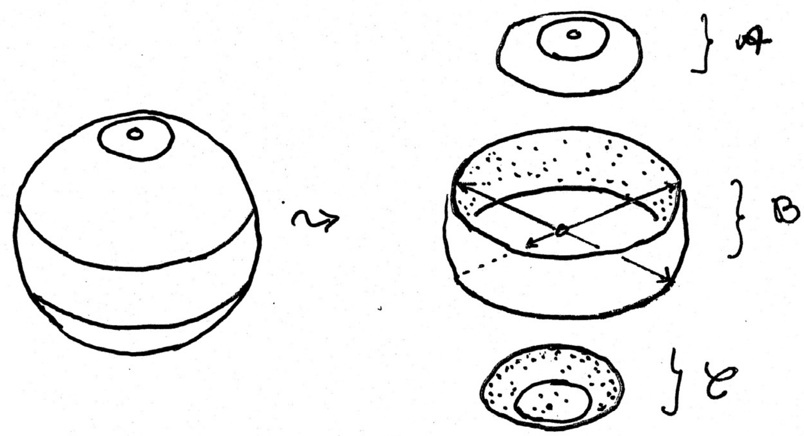

Fig. 5

Of the two lids, we delete the “southern” one *C* and keep *A.* Of the belt we still have to throw away one half, but have to be careful about identifying the ends. This way the belt becomes a Moebius strip *M* and the lid *A* has to be glued along the border of *M.* If we subdivide strip and lid into the triangles of Moebius (with the center of *A* marked F), then we are just back in his situation.

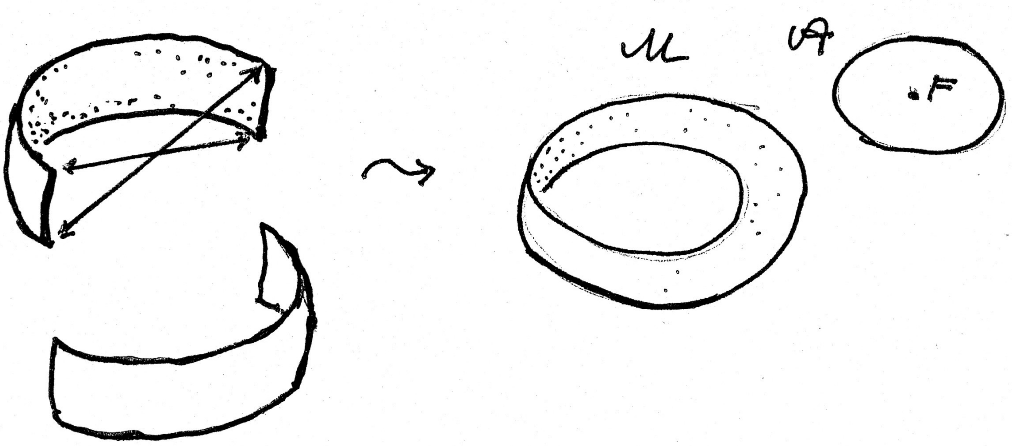

Fig. 6: Moebius strip and lid

The second more familiar way of representing P is by a cross-cap: throw away the “northern” hemisphere but take care of identifying opposite points on the equator right.

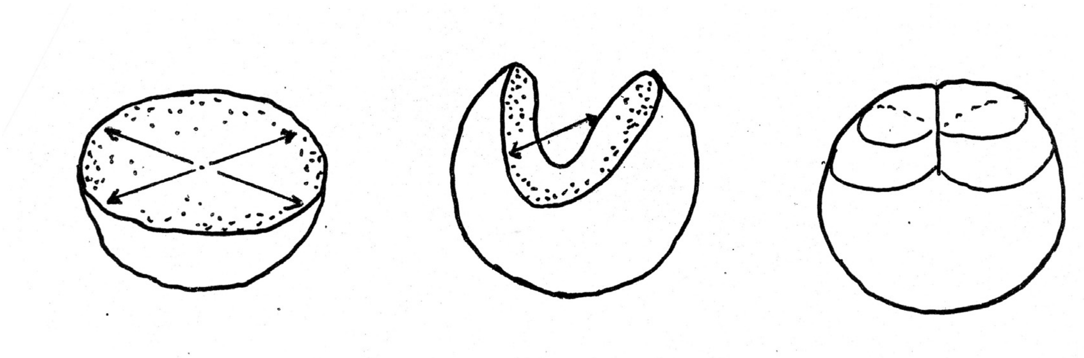

Fig. 7

4. Werner Boy[3]

Werner Boy (1879 - unknown) was a doctoral student of Hilbert, his published dissertation (Boy 1903) is about the (differential) topology of closed surfaces. He points out that the most well known “one sided” surface is the Klein bottle, shows it and points out that its (Euler-) characteristic is 0

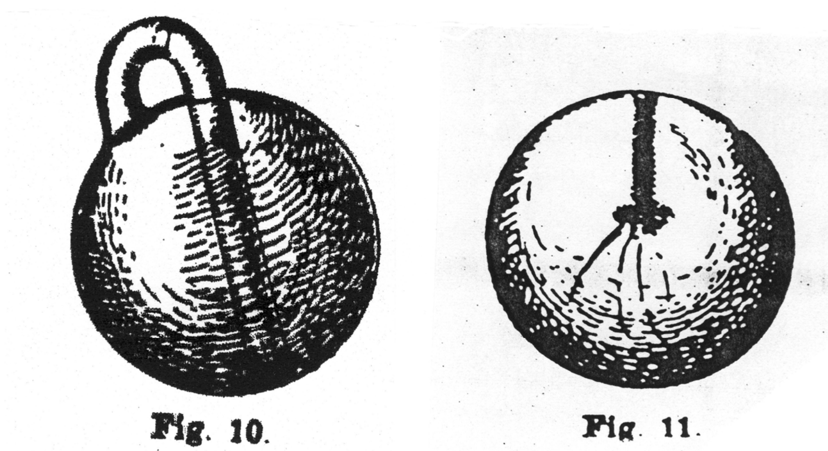

Fig. 8: The Klein bottle, Boy’s fig 10 and cross cap, his fig 11

[3] Apparently the last thing that is known about Werner Boy is, that he was a Gymnasium teacher at a school in Krefeld in 1908. The author is in contact with the people in the archives in Krefeld and hopes to find out more about Boy’s later life.

Next he mentions Steiner's Roman surface as an example of characteristic 1. Then he proceeds to a new picture, which is just our cross cap. Because he does not mention any earlier picture of the cross cap, he may have considered it to be well known. In fact, there is an earlier sketch of it and some more non-orientable surfaces by W. Dyck (1856 – 1934) in his (1888, figures in the appendix).

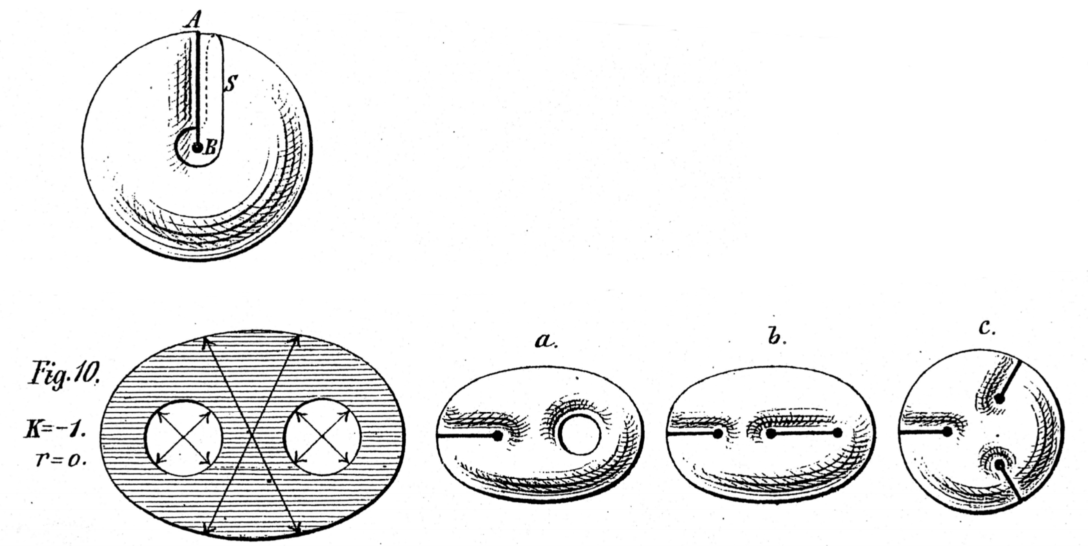

Fig. 9: Cross cap and other surfaces by Dyck 1888, plate 2

Unfortunately, the cross-cap has two pinch-points where it is not differentiable. Boy proceeds to construct an everywhere differentiable realization of *P*, the famous "Boy`s surface" and shows pictures of it from a front and reverse side:

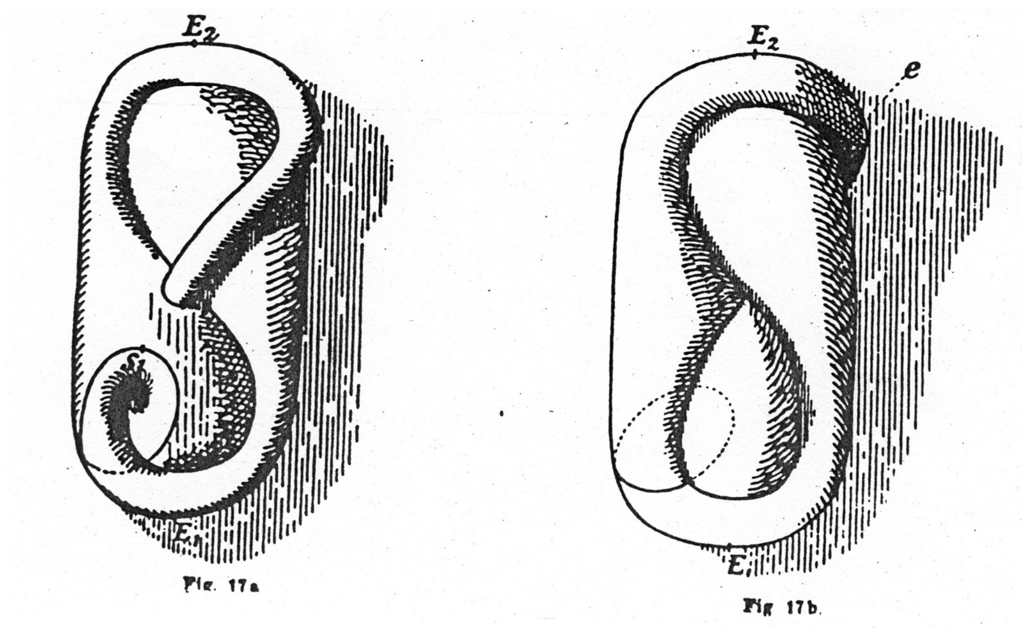

Fig. 10: Boy's figures 17 a,b

Plaster models of this can be found in the collections of many mathematical institutes, excellent photos of these models and related ones are in (Fischer I, 1986, p. 110-115). Boy himself draws still another view:

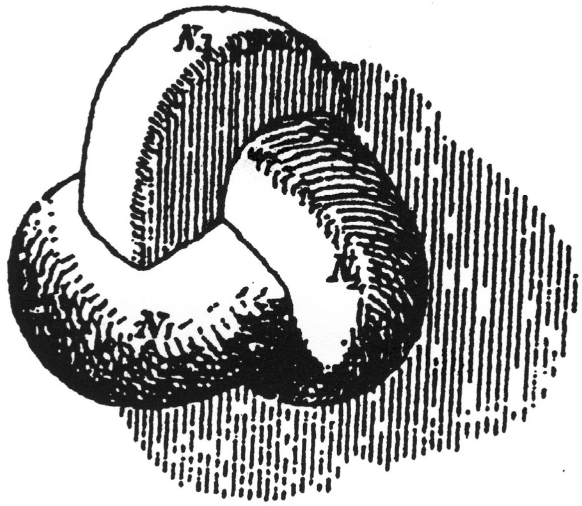

Fig. 20a.

Fig. 11: Boy`s figure 20a

From the pictures like Boy 17 a,b it is somewhat hard to see the internal structure of the surface. As a remedy for this, George Francis (1980 and 1987) had the great idea to cut suitable “windows” into the surface, and that way it is easy to see what is going on “inside” .

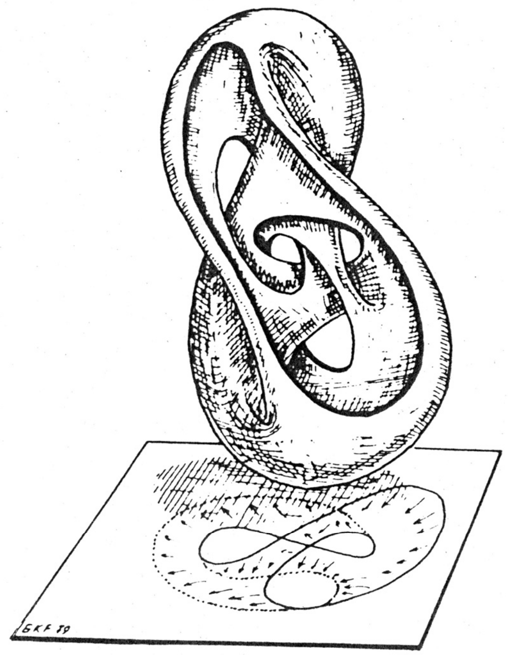

Fig. 12: Boy`s surface with 4 windows, G. Francis 1980, p.203

5. From Mathematics to Art

Here it is in order first to remember some statements concerning principal aspects of the relations of pictures in the sciences and the arts by the art-historian and philosopher G. Boehm (2001, 53):
Scientific pictures have their meaning outside themselves. In other words they are constructed to show something outside themselves, they are instruments, in contrast to works of art, which have their meaning in themselves. Scientific pictures intend a unique meaning, whereas works of art admit of different interpretations. Richness of metaphors typical for works of art is not useful in scientific pictures.

On the other hand, Boehm stresses that with the passage of time scientific pictures may loose their immediate purpose, and in the process of aging may just keep their aesthetic value and hence are transformed into works of art. Look at Boy`s and Dyck`s pictures Fig. 10 under this aspect! That means, in some senses at least, mathematical instruction is reconciled with aesthetical attraction. Keeping these considerations in mind we finally proceed to some works of art inspired by various forms of the projective plane, now just representing themselves and serving no other purpose.

Fig. 13: Ruth Vollmer: Steiner`s Roman Surface 1970[4]
Diam. 30,5 cm, Collection Dorothea and Leo Rabkin, New York.

[4] Photo Oliver Klasen. Copyright ZKM Karlsruhe.
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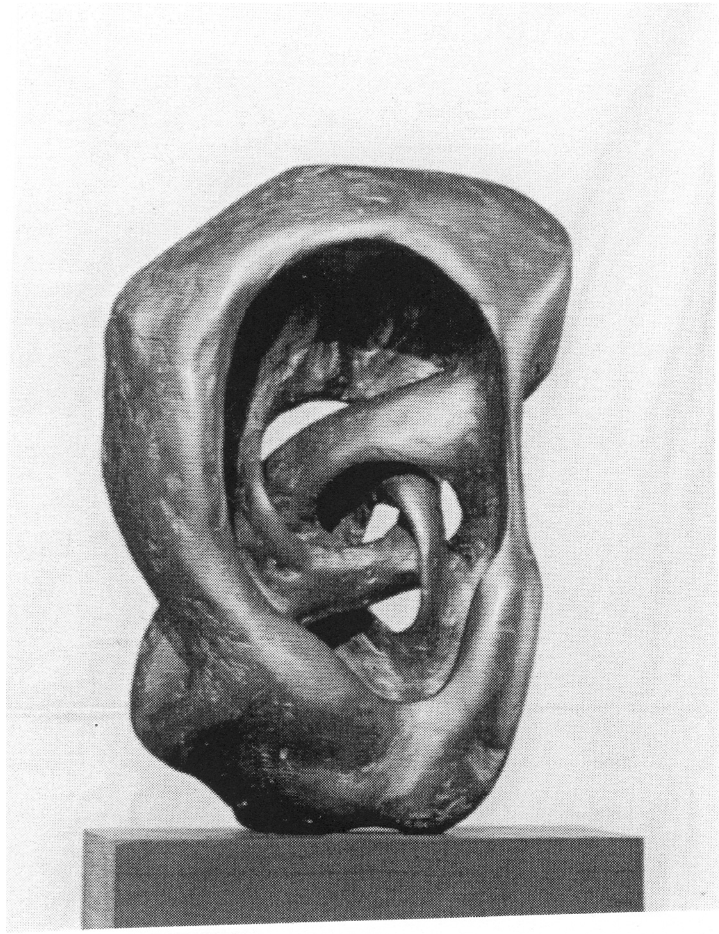

Fig. 14: Benno Artmann: Ich bin ganz Ohr 1982
Height 38 cm (Photo B. Artmann). Again Boy`s surface with 4 windows, inspired by G. Francis

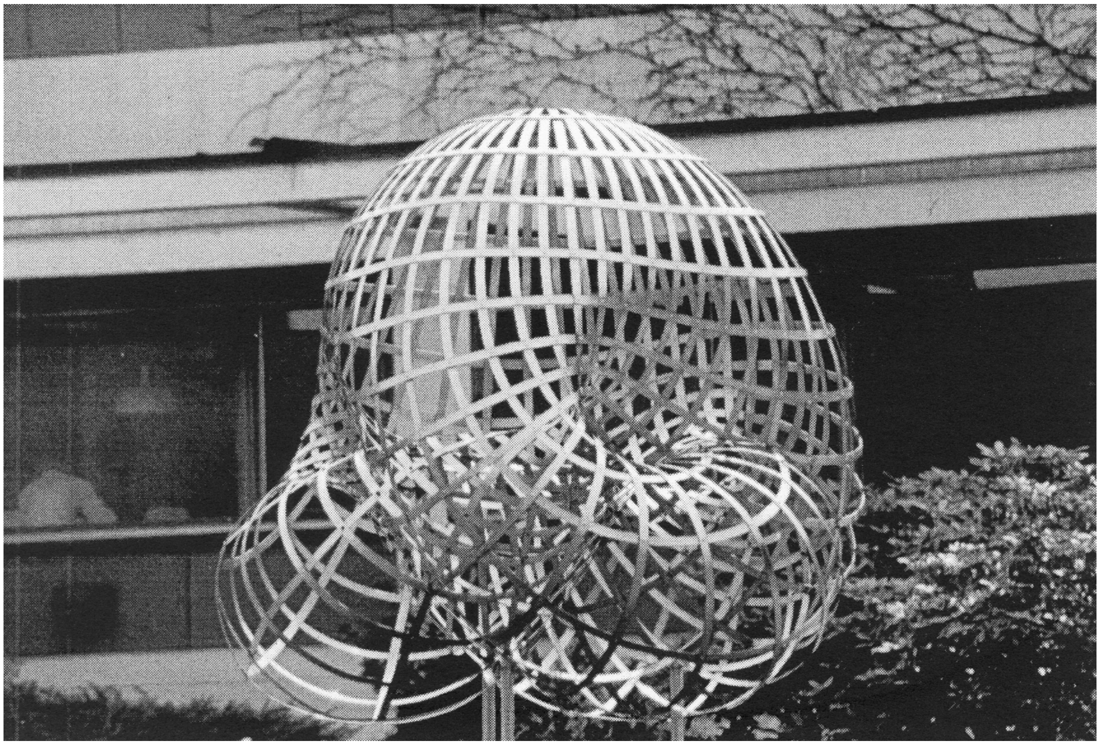

Fig 15: Boy`s surface[5] in Oberwolfach (Photo B. Artmann)

[5] The Boy-surface at Oberwolfach was conceived by Ulrich Pinkall in Berlin

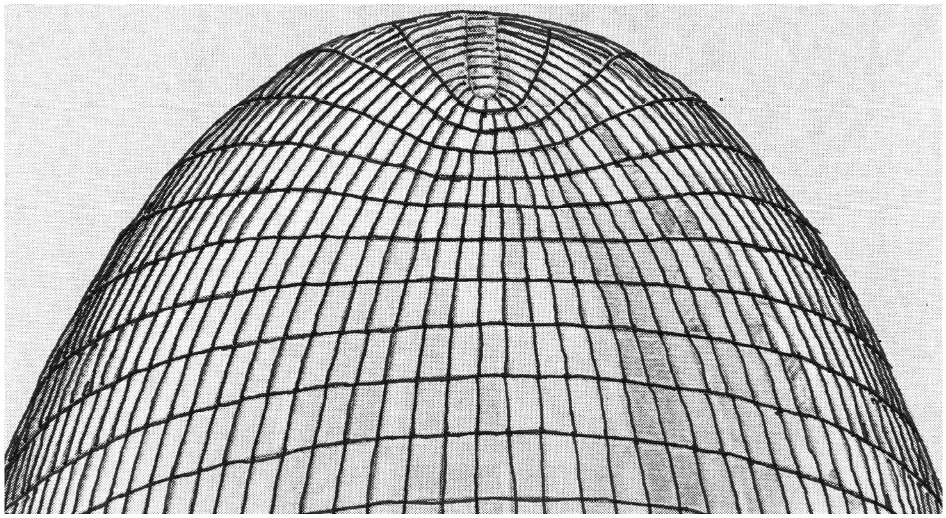

Fig. 16: The modern building of Peek and Cloppenburg Köln / Cologne 2005. Renzo Piano architects. Drawing after a photo by Fechner/Koch

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